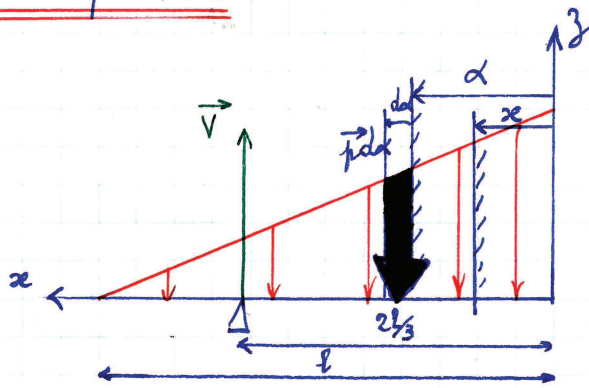


Si de symétrie de chargement et de géométrie, le diagramme de M est = / axe z ; pour T symétrie / centre de la poutre.

EX: 25 p 6-35



remarque : le solide à étudier présentant une symétrie de géométrie et de chgt nous ne traiterons que la partie gauche xi - à côté.

diagramme de T symétrique / centre poutre, M sera symétrique / axe de symétrie.

$$1) \begin{cases} x=0 & p=-h \\ x=l & p=0 \end{cases} \Rightarrow p = \frac{h}{l}x - ha = \frac{h}{l}(x-l)$$

Equation équilibre :  $\int_0^l p dx + V = 0 \Rightarrow$

$$\frac{h}{l} \int_0^l (x-l) dx + V = 0$$

$$\frac{h}{l} \left[ \frac{(x-l)^2}{2} \right]_0^l + V = 0$$

$$\Leftrightarrow -\frac{hl^2}{2} + \frac{hl}{l} + V = 0 \Rightarrow V = \frac{hl}{2}$$

2) Calcul de M et T

$$x \in (0, \frac{2l}{3}) \quad T = V + \int_x^l p dx = \frac{hl}{2} + \frac{h}{l} \int_x^l (x-l) dx$$

$$T = \frac{hl}{2} + \frac{h}{l} \left[ -\frac{(x-l)^2}{2} \right] = \frac{hl}{2l} [l^2 - x^2 + 2lx - l^2] = \frac{hx}{2l} (2l-x)$$

$$M = -V(\frac{2l}{3} - x) - \int_x^l p dx \cdot (x-x)$$

$\sum F$  force B de levier de la charge élémentaire.

$$M = -\frac{hl}{2} (\frac{2l}{3} - x) - \frac{h}{l} \int_x^l (x-l)(x-x) dx$$

$$(x^2 - x^2 - \alpha l + lx) dx = (x^2 - x(x+l) + lx) dx$$

$$= \left[ \frac{x^3}{3} - \frac{(x+l)x^2}{2} + lx^2 \right]_x^l = \frac{l^3 - x^3}{3} - \frac{(x+l)(l^2 - x^2)}{2} + lx(l-x)$$

$$\left[ \frac{x^3}{3} \right]_x^l =$$

on met en facteur:  $(l-x)$

$$\Rightarrow (l-x) \left[ \frac{1}{3} (l^2 + xl + lx^2) - \frac{(x+l)^2}{2} + lx \right]$$

$$= \frac{1}{6} (l-x) (-l^2 - x^2 + 2lx) = -\frac{(l-x)^3}{6}$$

$$\Rightarrow M = -\frac{hl}{6} (2l-3x) + \frac{h}{6l} (l-x)^3$$

ou

pour  $\alpha = x$   $p = \frac{h}{l} (x-l)$  force = aire triangle.

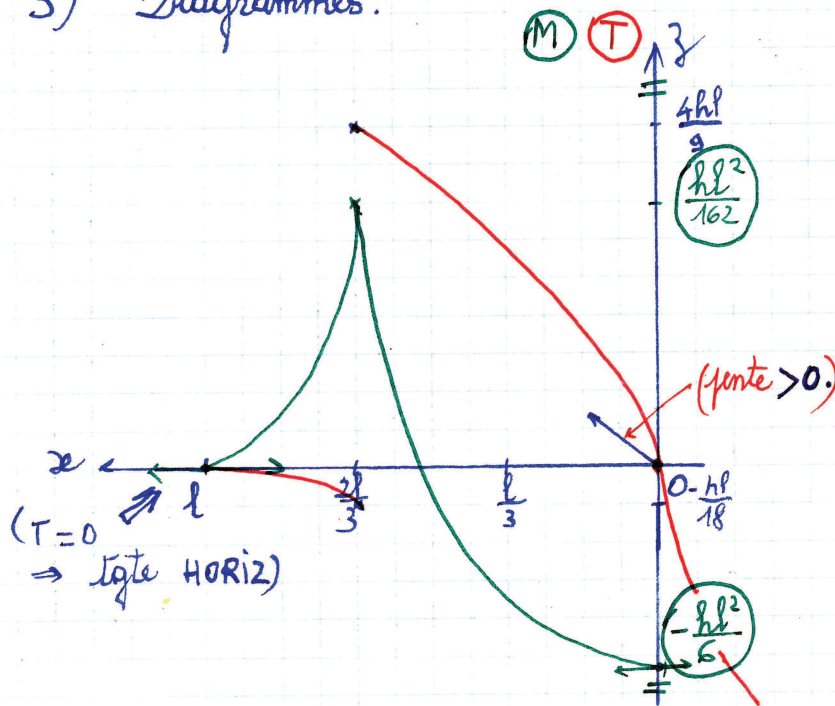
$$M = -\frac{hl}{6} (2l-3x) - \frac{\frac{h(l-x)^2(x-l)}{2l \times 3}}{3} = -\frac{hl}{6} (2l-3x) + \frac{hl}{6} (l-x)^3 + \frac{h}{6l} (l-x)^3$$

$$x \in \left(\frac{2l}{3}, l\right) \quad T = \int_x^l p dx = \int_x^l \frac{h}{l} (x-l) dx = -\frac{h}{2l} (x-l)^2$$

$$M = -\int_x^l p dx \times (l-x) = \frac{h}{6l} (l-x)^3$$

3) Diagrammes.

(M) (T)



$$\frac{dT}{dx} = \frac{h}{2l} (2l-x) - \frac{hx}{2l} = h - \frac{hx}{l}$$

$$x=0 \quad \frac{dT}{dx} = h \quad \text{pente} > 0$$

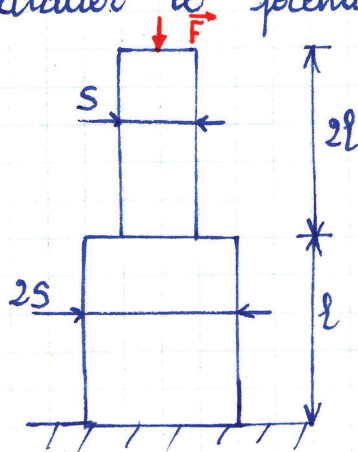
$$x = \frac{2l}{3} \quad \text{pour } x \in \left(\frac{2l}{3}, l\right)$$

$$T = -\frac{hl}{18}$$

$$x = \frac{2l}{3} \quad \text{pour } x \in \left(0, \frac{2l}{3}\right)$$

$$T = \frac{h}{3} \times \left(\frac{4l}{3}\right) = \frac{4hl}{9}$$

EX: Calculer le potentiel interne dans un pilier homogène. on néglige son propre poids.



$$\text{potentiel interne} = d\vec{\mathcal{E}}(\text{ext}) = -d\vec{\mathcal{E}}(\text{int}).$$

Calcul de N:

$$\text{en } G: \begin{cases} \vec{R} = \vec{F} \\ \mathcal{M}_G = \vec{0} \end{cases} \quad N = F.$$

Calcul du potentiel:

$$z \in [0, l] \quad S = 2S_0 \quad \mathcal{E} = \int_0^l \frac{1}{2} \frac{N^2}{ES} ds \Rightarrow \mathcal{E} = \int_0^l \frac{1}{2} \frac{N^2}{ES} dz.$$

$$z \in [l, 3l] \quad S = S_0$$

$$\Rightarrow \mathcal{E} = \frac{1}{2} \int_l^{3l} \frac{N^2}{ES} dz = \frac{1}{2} \left[ \int_0^l \frac{N^2}{2ES_0} dz + \int_l^{3l} \frac{N^2}{ES_0} dz \right]$$

$$\mathcal{E} = \frac{F^2}{2ES_0} \left[ \frac{l}{2} + 2l \right] = \frac{5F^2 l}{4ES_0}$$

$$\dim[\mathcal{E}] = \frac{\dim[F^2] \times \dim[l]}{\dim[E] \times \dim[S_0]}$$

$$\dim[\mathcal{E}] = \frac{M^2 L^2 T^{-4} L}{MLT^{-2} L^{-2} L^2} = ML^2 T^{-2}$$

$$\begin{cases} \text{vitesse} \equiv LT^{-1} \\ \text{accélération} \equiv LT^{-2} \\ F = ma \equiv MLT^{-2} \end{cases}$$

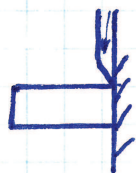
$$\tau = \frac{I}{I_y} \left( \frac{A}{e} \right)_z$$

$$\tau = G 2q \Rightarrow 2q = \frac{\tau}{G}$$

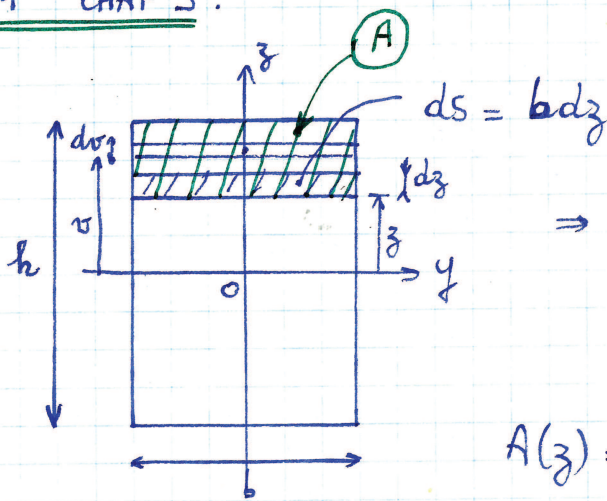
$$d\mathcal{E} = \int_s \frac{1}{2} \frac{\tau^2}{G} ds ds = \frac{1}{2} \left[ \int_s \frac{I^2}{I_y^2} \left( \frac{A}{e} \right)_z^2 \frac{1}{G} ds \right] ds$$

$$d\mathcal{E} = \left[ \frac{1}{2G} \frac{I^2}{I_y^2} \int_s \left( \frac{A}{e} \right)_z^2 ds \right] ds = \int_L \frac{1}{2G} \left[ \frac{I^2}{I_y^2} \int_s \left( \frac{A}{e} \right)_z^2 ds \right] ds$$

par identification avec  $\int_L \frac{1}{2} \frac{I^2}{GS} ds$  et  $2q = \frac{\tau}{G}$  valable  
seulement avec :



$$\Rightarrow S'_2 = \frac{I_y^2}{\int_s \left( \frac{A}{e} \right)_z^2 ds}$$

EX 1 CHAP 9.

$$\Rightarrow S'_z = \frac{I_y^2}{\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{A^2}{b} dz}$$

$$A(z) = \int_z^{\frac{h}{2}} b v dv$$

On a  $A(z) = \left(\frac{h}{2} - z\right) \frac{b}{2} \left(\frac{h}{2} + z\right) = \text{aire recte} \times \text{dist } 0 \text{ à CG (gravité de la surface)}$

$$A(z) = \frac{b}{8} (h^2 - 4z^2)$$

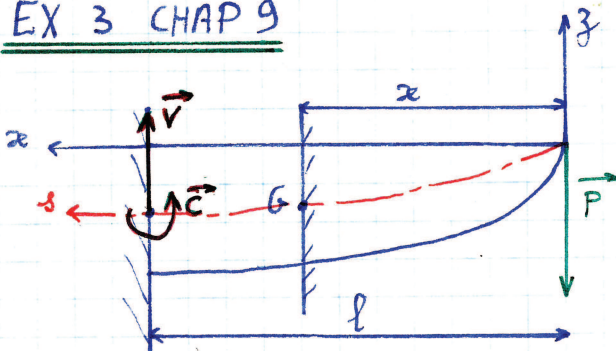
$$\Rightarrow \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{A^2}{b} dz = \frac{b}{64} \int_{-\frac{h}{2}}^{\frac{h}{2}} (h^2 - 4z^2)^2 dz$$

$$= \frac{b}{64} \left[ h^4 z + \frac{16z^5}{5} - 8h^2 \frac{z^3}{3} \right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= \frac{bh^5}{64} \left[ 1 + \frac{16}{5} \times \frac{1}{16} - \frac{8}{3} \times \frac{1}{4} \right] = \frac{bh^5}{64} \times \frac{8}{15} = \frac{bh^5}{120} = D(x)$$

d'où

$$S'_z = \frac{b^2 h^6}{144} \times \frac{120}{bh^5} = \frac{5}{6} bh = \frac{5}{6} S.$$

EX 3 CHAP 9

la rotation de flexion se fait / y  
 $\rightarrow I_y$ .

Comparaison de  $W(M)$  et  $W_T$ :

$$W_M = \frac{1}{2} \int_L \frac{M^2}{EI} ds$$

$$\sigma_{x \text{ maxi}} = R. \quad V_x \quad (\text{pour acier } R_e = R_c)$$

$$\sigma_{x \text{ maxi}} = \frac{M_{\text{maxi}} z_{\text{maxi}}}{I_y} = \frac{6 M l}{bh^3}$$

$$\text{On a } M = -Px. \Rightarrow \sigma_{x \text{ maxi}} = \frac{-6Px}{bh^2} = R$$

$$\Rightarrow h = \left( \frac{-6Px}{bR} \right)^{\frac{1}{2}}$$

On a:  $ds = dx$  (corps prismatique)  $L \gg e$



Donc 
$$W_M = \frac{1}{2} \int_0^l \frac{12 P^2 x^2 (bR)^{\frac{3}{2}}}{Eb (-6Px)^{\frac{3}{2}}} dx$$

$$W_M = \frac{P^{\frac{1}{2}} b^{\frac{1}{2}} R^{\frac{3}{2}}}{E 6^{\frac{3}{2}}} \left[ \frac{2x^{\frac{3}{2}}}{3} \right]_0^l = \frac{2}{3E} \left( \frac{-P b R^3 l^3}{6} \right)^{\frac{1}{2}} = W_M$$

$$W_T = \frac{1}{2} \int \frac{T^2}{GSr} ds \quad \text{avec} \quad S'_r = \frac{5}{6} S \quad (\text{sect rect})$$

$$W_T = \frac{1}{2} \int \frac{6P^2 \sqrt{6R}}{G \cdot 5b (-6Px)^{\frac{3}{2}}} = \frac{-3 P^{\frac{3}{2}} R^{\frac{1}{2}}}{5G b^{\frac{1}{2}}} \int_0^l x^{-\frac{3}{2}} dx$$

$$W_T = \frac{-3 P^{\frac{3}{2}} R^{\frac{1}{2}} 2l^{\frac{1}{2}}}{5G b^{\frac{1}{2}} 6^{\frac{1}{2}}} = \frac{6}{5G} \left( \frac{-P^3 R l}{6b} \right)^{\frac{1}{2}} = W_T$$

Comparaison:

$$\frac{W_M}{W_T} = \frac{10G (R^3 b^2 l^3 6)^{\frac{1}{2}}}{18E (6R^3 R l)^{\frac{1}{2}}} = \frac{5G (b^2 R^2 l^2)^{\frac{1}{2}}}{9EP} = \frac{5G b R l}{9EP}$$

EX:  $l = 1m$  ;  $R = 12 \text{ daN/mm}^2$  ;  $E = 20\,000 \text{ daN/mm}^2$

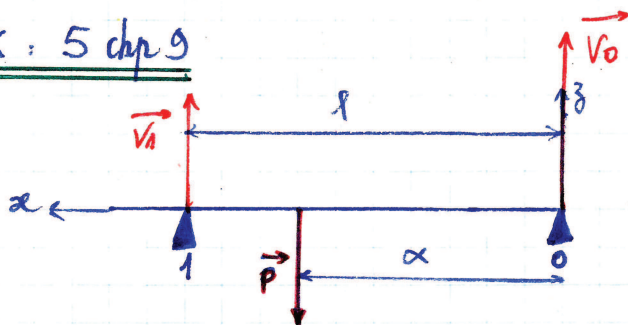
$G = \frac{2}{5} E$  (pour matériaux métallique)  $\rightarrow G = 8000 \text{ daN/mm}^2$

$P_0 = 800 \text{ daN}$  ;  $b = 50 \text{ mm}^2$

$$\frac{W_M}{W_T} = \frac{5 \cdot 8000 \cdot 50 \cdot 12 \cdot 1000}{9 \cdot 20\,000 \cdot 800} = 166,6 = \frac{W_M}{W_T}$$

On peut négliger  $W_T$  devant  $W_M$ .

EX: 5 chp 9



$$G = \frac{2}{5} E \quad \alpha = \frac{l}{3}$$

$$M_{t/0} \text{ et } M_{t/1} \rightarrow \left[ V_1 = -\frac{P}{3} \text{ et } V_0 = -\frac{2P}{3} \right]$$

Calcul de M et T

$$x \in (0, \frac{l}{3}) \quad \begin{cases} M = -V_0 x = \frac{2P}{3} x \\ T = \frac{2P}{3} \end{cases}$$

$$x \in (\frac{l}{3}, l) \quad \begin{cases} M = -V_1 (l-x) = \frac{P}{3} (l-x) \\ T = -\frac{P}{3} \end{cases}$$

Calcul de  $W_T, W_M$ .

$$W_T = \frac{1}{2} \int_L \frac{T^2}{GS'_2} ds = \frac{1}{2GS'_2} \left[ \frac{4P^2}{9} \int_0^{\frac{l}{3}} dx + \frac{P^2}{9} \int_{\frac{l}{3}}^l dx \right]$$

$$W_T = \frac{P^2 l}{9GS'_2}$$

$$W_M = \frac{1}{2} \int_L \frac{M^2}{EI} ds = \frac{1}{2EI} \left[ \frac{4P^2}{9} \int_0^{\frac{l}{3}} x^2 dx + \frac{P^2}{9} \int_{\frac{l}{3}}^l (l-x)^2 dx \right]$$

$$W_M = \frac{P^2}{18EI_y} \left[ \frac{4l^3}{81} + \frac{8l^3}{81} \right]$$

$$W_M = \frac{2P^2 l^3}{243EI_y}$$

Comparaison  $W_M$  et  $W_T$ .

$$\frac{W_M}{W_T} = \frac{2l^2 GS'_2}{27EI_y}$$

$$\begin{cases} I_y = \frac{bh^3}{12} \\ G = \frac{2}{5} E \\ S'_2 = \frac{5}{6} bh \text{ (rect)} \end{cases}$$

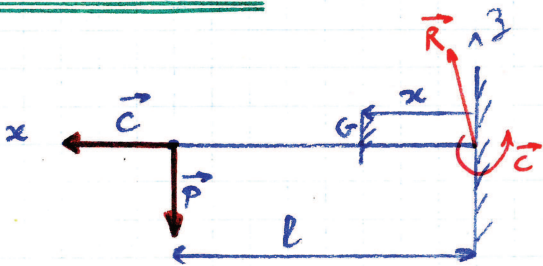
EX 6 CHAP 9.

$$\frac{W_M}{W_T} = \frac{8l^2}{27h^2}$$

$$l = 10h$$

$$\Rightarrow \frac{W_M}{W_T} = 29,62.$$

EX 9 CHAP 9



$$\begin{cases} U = -Pl \\ T = P \\ M = -P(l-x) \end{cases}$$

Calcul des  $\neq W$ .

$$W_U = \frac{1}{2} \int_L \frac{U^2}{GJ_1} dx = \frac{1 \times 2 \times P^2 \times l^2}{26\pi r^4} \int_0^l dx = \frac{P^2 l^3}{6\pi r^4}$$

$$W_T = \frac{1}{2} \int_L \frac{T^2}{GS'_2} ds = \frac{5P^2 l}{96\pi r^2}$$

$$W_M = \frac{1}{2} \int_L \frac{M^2}{EI} ds = \frac{2P^2}{\pi r^4 E} \left[ \frac{(l-x)^3}{-3} \right]_0^l = \frac{2P^2 l^3}{3\pi r^4 E}$$

Comparaisons :

$$\frac{W_U}{W_T} = \frac{9}{5} \left( \frac{l}{r} \right)^2 \quad \text{si } \frac{l}{r} = 10 \quad \frac{W_U}{W_T} = 180$$

$$\frac{W_u}{W_m} = \frac{15}{4} = 3,75.$$

On peut négliger  $W_T$  devant les autres.

th travaux virtuel:

- (H) { Système en équilibre  
dft virtuel  
infinitement petit  
compatible avec les liaisons.

(C)  $\Sigma W_{Fext} + \Sigma W_{Fint} = 0.$

th FONVIOLANT: application du théorème des travaux virtuel à un syst isostatique associé.

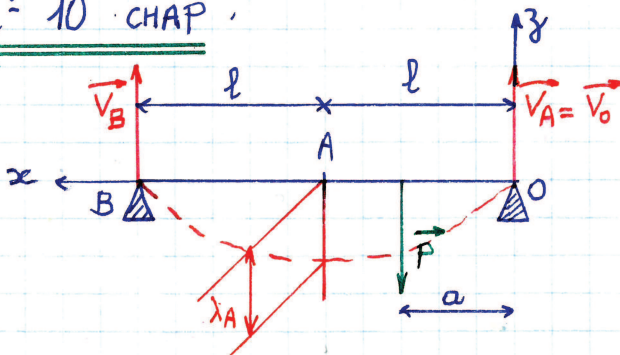
Application de FONVIOLANT:

- (1) déf du SIA. { \* géométriquement identique  
\* liaison = liaison syst réel - k (° hyperst.)  
\* ttes les  $\vec{F}_{ext}$  du SR sont supprimés  
\* mise en place de F et C (auxiliaire)

- (2) déf du dft virtuel  
→ déformation virtuelle = déf réelle du SR  
\* là où on a supprimé les liaisons. (calcul des réactions)  
\* à l'endroit où l'on veut calculer les dfts.

$$\Sigma F_i \lambda_i + \Sigma C_j \varphi_j = W'$$

EX m° 10 CHAP.



Calcul de  $V_0$  et  $V_B$

$M_{t/O}$  et  $M_{t/B}$

$$V_B = \frac{-Pa}{2l}$$

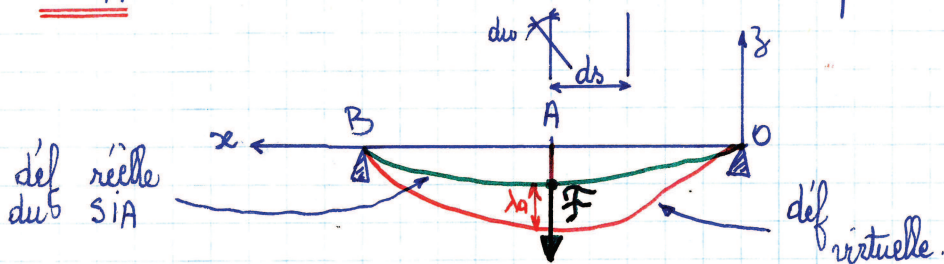
$$V_0 = \frac{-P(2l-a)}{2l}$$

Calcul du moment de Flexion:

$$x \in (0, a) \quad M = -V_0 \cdot x = \frac{P(2l-a)x}{2l}$$

$$x \in (a, 2l) \quad M = -V_B(2l-x) = \frac{Pa(2l-x)}{2l}$$

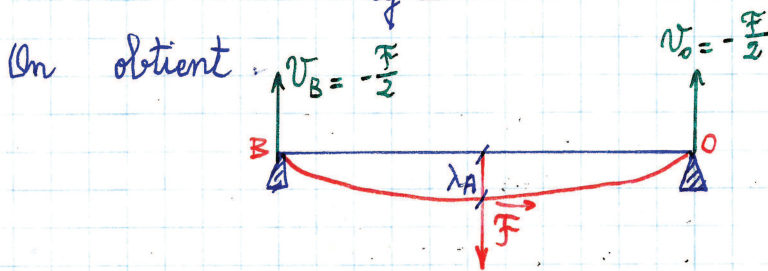
SIA en dessous du SR et en correspondance.



On place les forces (inconnues) où l'on veut calculer le dpt (F)

on a  $dw = K ds = \frac{M}{EI} ds$

On a  $F \lambda_A = \int_L M \frac{M}{EI} ds = \text{travail de } F$



on prends la déformation réelle du SIA petit  $\rightarrow$  confondu avec BO

$$x \in (0, l) \quad M = -\left[-x \frac{F}{2}\right] = x \frac{F}{2}$$

$$x \in (l, 2l) \quad M = \frac{Fl}{2} (2l-x)$$

Application de Formule de Betti : ( $a < l$ )

$$F \lambda_A = \int_L M \frac{M}{EI} ds$$

$$EI F \lambda_A = \frac{FP(2l-a)}{4l} \int_0^a x^2 dx + \frac{FPa}{4l} \int_a^l (2l-x)x dx + \frac{FPa}{4l} \int_l^{2l} (2l-x)^2 dx$$

et ceci  $\forall F$ .

(suppression de F)

$$EI \lambda_A = \frac{P}{4l} \left[ (2l-a) \frac{a^3}{3} + a \left( \frac{2l^3}{2} - \frac{2la^2}{2} - \frac{l^3}{3} + \frac{a^3}{3} \right) + \frac{al^3}{3} \right]$$

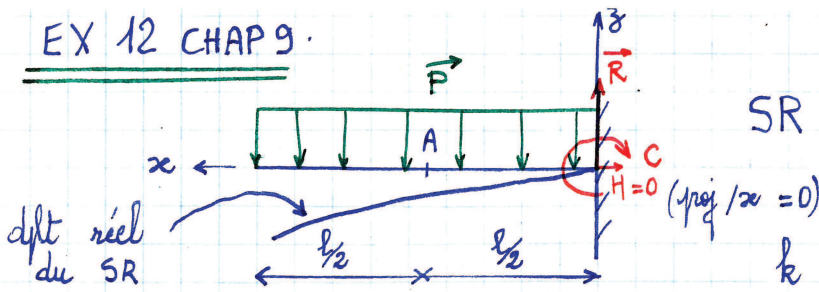
du 4<sup>ème</sup> degré  $\rightarrow$  correct.

$$EI \lambda_A = \frac{1}{3} \frac{Pa}{4l} \left( -\frac{3la^2}{3} + \frac{4l^3}{6} - \frac{3l^3}{3} \right) = \frac{Pa}{12} (l^2 - a^2)$$

$$\lambda_A = \frac{Pa}{12EI} (3l^2 - a^2)$$



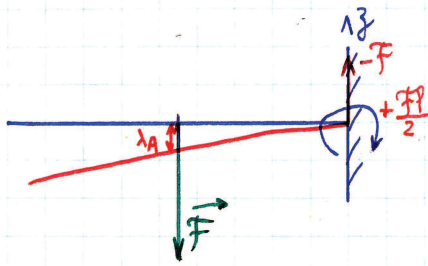
EX 12 CHAP 9.



dft réel du SR

SR  
3 équations dans le plan  
6 équations dans l'espace.  
 $k = \text{nbre équation} + \text{nbre inconnus}$

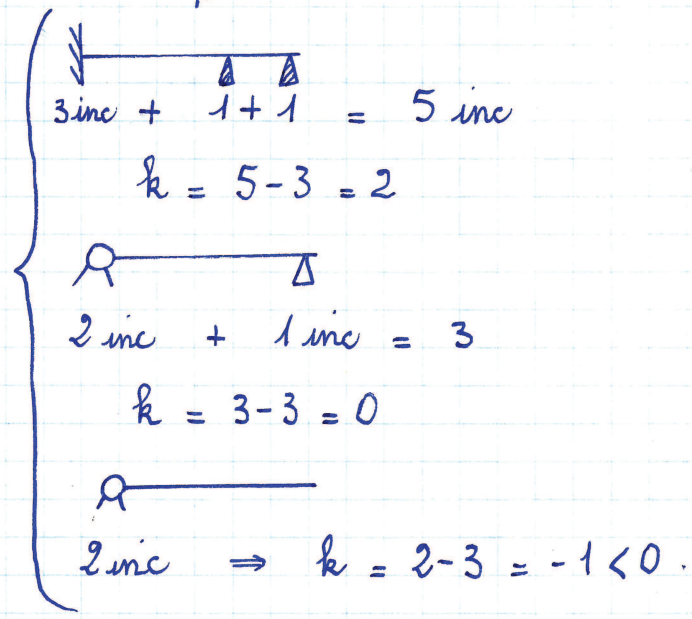
SIA



$\left\{ \begin{array}{l} > 0 \text{ hyperstatique} \\ = 0 \text{ isostatique} \\ < 0 \text{ cinématique, mécanisme, hypostatique} \end{array} \right.$

dft virtuel (SIA) = dft réel (SR)

Couple à l'encastrement  $C = +\frac{Fl}{2}$   
négatif (sens) or  $F < 0 \Rightarrow +\frac{Fl}{2}$



Calcul de M.

$x \in (0, l)$   $M = \frac{-p(l-x)^2}{2}$

Calcul de M

$x \in (0, \frac{l}{2})$   $M = -F(\frac{l}{2} - x)$

$x \in (\frac{l}{2}, l)$   $M = 0$

th de Betti

$F\lambda_A = \int_L \frac{MM}{EI} dx \quad VF$

$EIF\lambda_A = \int_0^{\frac{l}{2}} -F(\frac{l}{2} - x) \cdot \frac{(-p)}{2} (l-x)^2 dx$

$EIF\lambda_A = \frac{FP}{2} \int_0^{\frac{l}{2}} (\frac{l}{2} - x)(l-x)^2 dx \quad (\text{vu } \frac{l}{2} - x = l - x - \frac{l}{2})$

$VF \Rightarrow EIF\lambda_A = \frac{P}{2} \int_0^{\frac{l}{2}} (l-x)^3 dx - \int_0^{\frac{l}{2}} \frac{1}{2} (l-x)^2 dx$

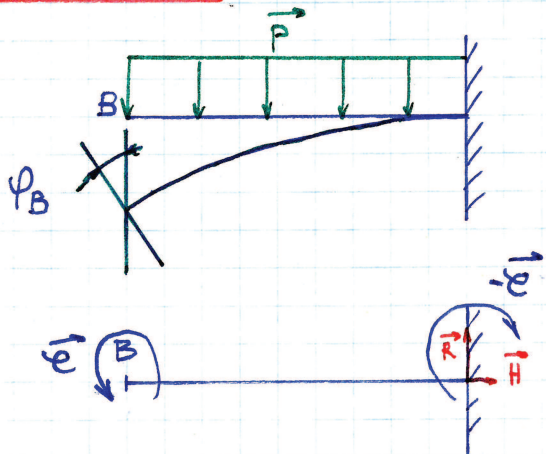
$$EI \lambda_A = \frac{P}{2} \left[ -\frac{(l-x)^4}{4} \right]_0^l - \frac{P}{2} \left[ -\frac{(l-x)^3}{3} \right]_0^l$$

$$EI \lambda_A = \frac{P}{2} \left[ -\frac{l^4}{64} + \frac{l^4}{4} - \frac{P}{2} \left[ -\frac{l^3}{24} + \frac{l^3}{3} \right] \right]$$

$$EI \lambda_A = \frac{Pl^4}{2} \left[ \frac{15}{64} - \frac{7}{48} \right] = \frac{17Pl^4}{384}$$

$$\Rightarrow \lambda_A = \frac{17Pl^4}{384EI}$$

Calcul de  $\varphi_B$



SR

pas de R et H (proj / x = H = 0)  
proj / z = R = 0

$$x \in (0, l) \quad m = P$$

$$\text{On a } \varphi_B = \int_L \frac{Mm}{EI} dx$$

$$\varphi_B = \left[ \int_0^l (l-x)^2 dx \right] \cdot \frac{-Pl}{2EI} \quad \forall P$$

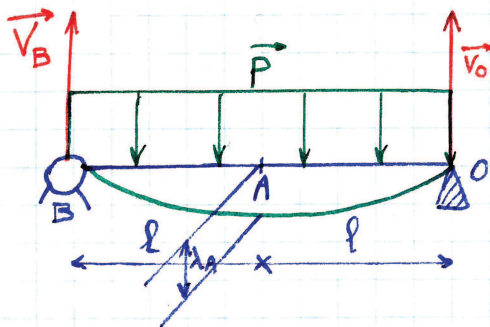
$$\Rightarrow EI \varphi_B = -\frac{P}{6} \left[ -(l-x)^3 \right]_0^l = \frac{-Pl^3}{6}$$

$$\Rightarrow \varphi_B = \frac{-Pl^3}{6EI}$$

( $\varphi_B$  en radian [sans dimension])

$$\dim \varphi_B = \frac{Nm^{-1} m^3}{Nm^{-2} m^4} = 1 = \frac{\overbrace{MLT^{-2}}^{Force} L^{-1} L^3}{MLT^{-2} L^2 L^4} = 1$$

EX:



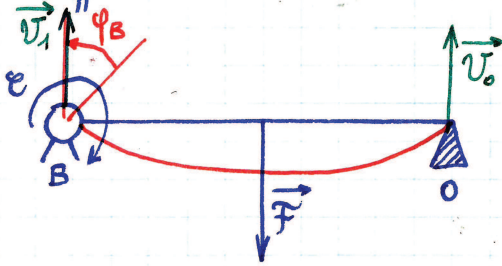
## Calcul des actions associées de contacts:

$$V_0 = V_B = -Pl$$

## Calcul de $\lambda_A$ et $\varphi_B$ :

par application de Fermirolant:

$$F \lambda_A + e \varphi_B = W' = \int_0^l \frac{mM}{EI} ds$$



$$x \in (0, 2l) \quad M = \frac{px(2l-x)}{2}$$

## Calcul des actions de contact dans le SIA.

$$V_0 + V_1 + F = 0$$

$$m_{t/B} = 2lV_0 + Fl + e = 0$$

$$V_0 = \frac{-e - Fl}{2l} = -\frac{e}{2l} - \frac{F}{2}$$

$$V_1 = -F + \frac{e}{2l} + \frac{F}{2}$$

$$x \in (0, l) \quad \mathcal{N} = -(V_0 x) = \left(\frac{e}{2l} + \frac{F}{2}\right)x$$

$$x \in (l, 2l) \quad \mathcal{N} = -[V_0 x + F(x-l)] = \left(\frac{e}{2l} + \frac{F}{2}\right)x - F(x-l)$$

D'où :

$$\frac{P}{2} \int_0^l x^2 (2l-x) \left(\frac{e}{2l} + \frac{F}{2}\right) dx + \frac{P}{2} \int_l^{2l} x^2 (2l-x) \left(\frac{e}{2l} + \frac{F}{2}\right) dx - \frac{P}{2} \dots$$

$$\dots \int_l^{2l} x(x-l)(2l-x) F dx$$

$$= \frac{e}{2l} \frac{P}{2} \underbrace{\int_0^{2l} x^2 (2l-x) dx}_{I_1} + \frac{FP}{2} \left[ \frac{1}{2} \int_0^{2l} x^2 (2l-x) dx - \underbrace{\int_l^{2l} x(x-l)(2l-x) dx}_{I_2} \right]$$

$$I_1 = \left[ 2l \frac{x^3}{3} - \frac{x^4}{4} \right]_0^{2l} = \frac{4l^4}{3}$$

$$I_2 = \int_l^{2l} (x^2 - lx)(2l-x) dx = \int_l^{2l} (2lx^2 - x^3 - 2lx^2 + lx^2) dx$$

$$I_2 = l^4 \left[ 7 - 3 - \frac{15}{4} \right] = \frac{l^4}{4}$$

Donc :

$$F \lambda_A + \mathcal{E} \varphi_B = \frac{\mathcal{E}}{2\mathcal{E}} \frac{P}{2EI} \frac{4l^4}{3} + \frac{F}{2} \frac{P}{EI} \left( \frac{2l^4}{3} - \frac{l^4}{4} \right)$$

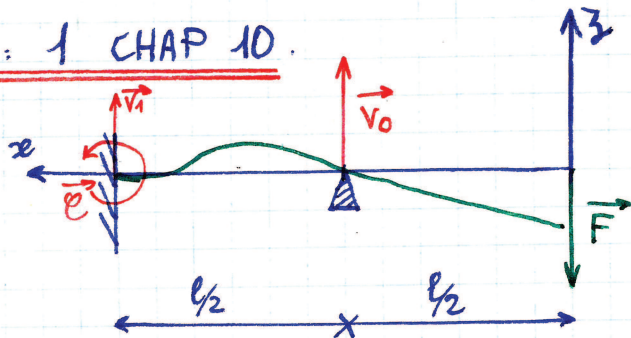
par identification :

$$\lambda_A = \frac{5Pl^4}{24EI}$$

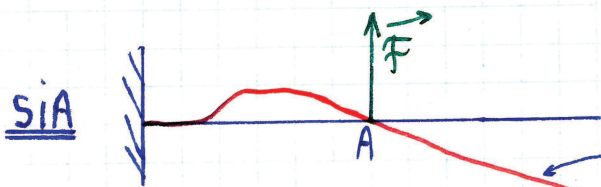
et

$$\varphi_B = \frac{Pl^3}{3EI}$$

EX : 1 CHAP 10.



$$EI = c^{\text{te}}$$



SIA

def virtuelle = def réelle SR.

th de Fermirolant :

$$F \lambda_A = \int \frac{\mathcal{M}}{EI} ds = 0 \quad (\lambda_A = 0)$$

$$\text{comme } EI = c^{\text{te}} \Rightarrow \int \mathcal{M} = 0$$

Calcul de M

$$x \in (0, \frac{l}{2}) \quad M = -Fx$$

$$x \in (\frac{l}{2}, l) \quad M = -Fx - V_0(x - \frac{l}{2})$$

Calcul de \mathcal{L}

$$x \in (0, \frac{l}{2}) \quad \mathcal{L} = 0$$

$$x \in (\frac{l}{2}, l) \quad \mathcal{L} = -F(x - \frac{l}{2})$$

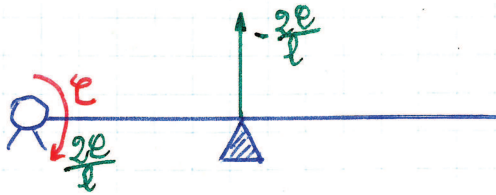
$$\text{On a : } F \int_{\frac{l}{2}}^l [Fx + V_0(x - \frac{l}{2})] (x - \frac{l}{2}) dx = 0 \quad \forall F$$

$$\int_{\frac{l}{2}}^l [F(x^2 - \frac{lx}{2}) + V_0(x - \frac{l}{2})^2] dx = F l^3 \left( \frac{7}{24} - \frac{3}{16} \right) + V_0 l^3 \left( \frac{1}{24} \right)$$

$$\Rightarrow 5F + 2V_0 = 0$$

$$\Rightarrow V_0 = -\frac{5}{2}F.$$

$$\left. \begin{array}{l} V_0 + V_1 + F = 0 \\ 2F + V_0 \frac{l}{2} + C_1 = 0 \end{array} \right\} \Rightarrow C_1 \text{ et } V_1.$$



On remplace l'encastrement par une rotule.

$$x \in (0, \frac{l}{2}) \quad M = -Fx$$

$$x \in (\frac{l}{2}, l) \quad M = -Fx - V_0(x - \frac{l}{2})$$

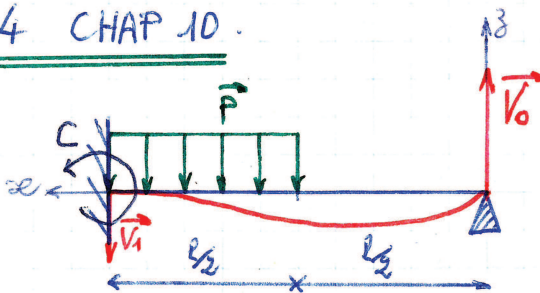
$$x \in (0, \frac{l}{2}) \quad M = 0$$

$$x \in (\frac{l}{2}, l) \quad M = \frac{2\ell}{l} \left( \frac{2x-l}{2} \right)$$

$$\Rightarrow -\frac{2\ell}{l} \int_{\frac{l}{2}}^l \left[ F(x^2 - \frac{l}{2}x) + V_0(x - \frac{l}{2})^2 \right] dx = 0 \quad \forall \ell$$

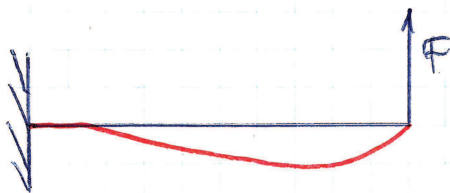
On retrouve la même équation que précédemment.

#### EX 4 CHAP 10.



la poutre est homogène et de section constante  $\Rightarrow EI = c^{te}$

SIA



on a donc :

$$\int_l \mathcal{M} M ds = 0.$$

$$\left\{ \begin{array}{l} x \in (0, \frac{l}{2}) \quad M = -V_0 x \\ x \in (\frac{l}{2}, l) \quad M = -V_0 x - \frac{F}{2} (x - \frac{l}{2})^2 \end{array} \right.$$

$$\left\{ \begin{array}{l} x \in (0, \frac{l}{2}) \quad M = -V_0 x \\ x \in (\frac{l}{2}, l) \quad M = -V_0 x - \frac{F}{2} (x - \frac{l}{2})^2 \end{array} \right.$$

$$x \in (0, l) \quad \mathcal{M} = -Fx.$$

Résolution :  $\int_0^l \mathcal{M} M dx = 0$

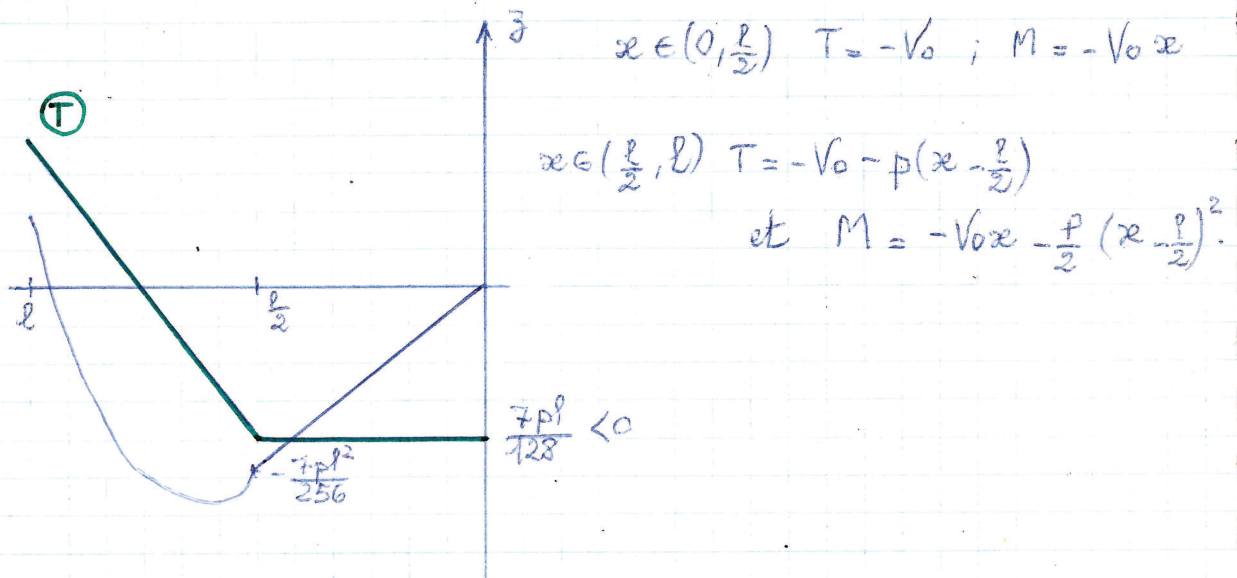
$$\Rightarrow \int_0^{\frac{l}{2}} V_0 x^2 dx + \frac{F}{2} \int_{\frac{l}{2}}^l x (x - \frac{l}{2})^2 dx = 0$$

$$\frac{Vol^3}{3} + \frac{P}{2} \left[ \frac{x^4}{4} - \frac{lx^3}{3} + \frac{l^2 x^2}{8} \right] \frac{l}{2} = 0$$

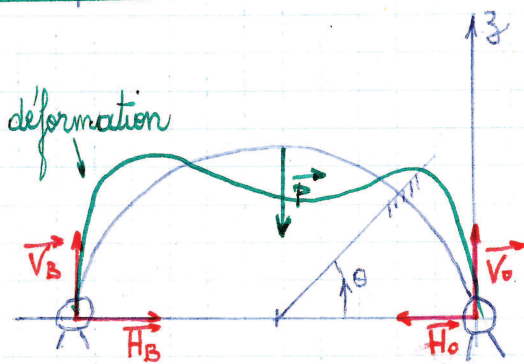
$$\Rightarrow \frac{Vol^3}{3} + \frac{P}{2} \left( \frac{8}{192} - \frac{1}{192} \right) = 0$$

$$\text{Donc } V_0 = \frac{-7Pl}{128}$$

### Diagramme de M et T



### EX n° 13 chap 10.



la statique donne

$$V_0 = V_B = \frac{-P}{2} \text{ (symétrie).}$$

$$H_0 = -H_B$$

Calcul de M:  $M = -[-H_0 R \sin \theta + V_0 R (1 - \cos \theta)]$

$$M = +H_0 R \sin \theta - V_0 R (1 - \cos \theta) \quad \theta \in (0, \frac{\pi}{2})$$

th de clébrée:

$$\frac{\partial \mathcal{E}}{\partial H_0} = 0$$

On a :  $\mathcal{E} = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{M^2}{EI} ds \Rightarrow \frac{\partial \mathcal{E}}{\partial H} = \frac{1 \cdot 2}{2EI} \int_0^{\frac{\pi}{2}} M \frac{\partial M}{\partial H} ds = 0$

$$2 \times \int_0^{\frac{\pi}{2}} [H_0 R \sin \theta - V_0 R (1 - \cos \theta)] R \sin \theta R d\theta = 0 \quad \text{on prends 2}$$

car la poutre est symétrique, l'énergie élastique dans les 2 moitiés d'arcs

sont égales  $\Rightarrow \int_0^{\frac{\pi}{2}} H_0 \sin^2 \theta d\theta - \int_0^{\frac{\pi}{2}} V_0 \sin \theta d\theta + \int_0^{\frac{\pi}{2}} V_0 \cos \theta \sin \theta d\theta = 0$

$$\int_0^{\frac{\pi}{2}} H_0 \sin^2 \theta d\theta = \int_0^{\frac{\pi}{2}} H_0 \frac{1 - \cos 2\theta}{2} d\theta = \frac{H_0}{2} \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}} = \frac{H_0 \pi}{2} \times \frac{1}{2}$$

$$\int_0^{\frac{\pi}{2}} V_0 \sin \theta d\theta = V_0 [-\cos \theta]_0^{\frac{\pi}{2}} = 2V_0 \times \frac{1}{2} = V_0$$

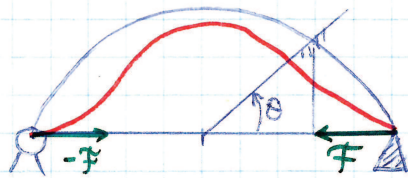
$$\int_0^{\frac{\pi}{2}} \frac{V_0}{\sin 2} \sin 2\theta d\theta = \frac{V_0}{2} \left[ -\frac{1}{2} \cos 2\theta \right]_0^{\frac{\pi}{2}} = \frac{-V_0}{2} \left[ -\frac{1}{2} + \frac{1}{2} \right] = \frac{V_0}{2}$$

D'où :

$$\frac{H_0 \pi}{4} - V_0 + \frac{V_0}{2} = 0 \Rightarrow H_0 = \frac{2V_0}{\pi} = \frac{-P}{\pi}$$

Calcul par fémulation :

SIA :



là où on veut calculer la force.

$$\theta \in (0, \pi) \quad \mathcal{K} = -FR \sin \theta$$

$$F\lambda = W' \quad (\text{potentiel interne virtuel})$$

et  $W' = 2W'$  (sur les  $\frac{1}{2}$  parties : symétries).

$$\text{D'où : } 0 = 2 \int_0^{\frac{\pi}{2}} \frac{\mathcal{K}M}{EI} ds$$

$$\text{D'où } \int_0^{\frac{\pi}{2}} \mathcal{K}M ds = 0 = -F \int_0^{\frac{\pi}{2}} [H_0 R \sin \theta - V_0 R (1 - \cos \theta)] R \sin \theta R d\theta$$

$\forall F$

On retrouve la même équation que précédemment.

Calcul de  $\lambda_A$  :

Castigliano

$$\lambda_A = \frac{JW}{JF}$$

$$W = \frac{1}{2} \int \frac{M^2}{EI} ds = 2 \times \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{M^2}{EI} ds$$

$$\Rightarrow W \cdot EI = \int_0^{\frac{\pi}{2}} M^2 R d\theta = \int_0^{\frac{\pi}{2}} [H_0 R \sin \theta + V_0 R (1 - \cos \theta) - 2H_0 V_0 R \sin \theta (1 - \cos \theta)]^2 R d\theta$$

$$\Rightarrow \frac{EIW}{R^3} = H_0^2 \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta + V_0^2 \int_0^{\frac{\pi}{2}} (1 - 2\cos \theta + \cos^2 \theta) d\theta -$$

$$\rightarrow 2 H_0 V_0 \int_0^{\frac{\pi}{2}} (\sin \theta - \sin \theta \cos \theta) d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2} d\theta = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} 2 \cos \theta d\theta = [2 \sin \theta]_0^{\frac{\pi}{2}} = 2$$

$$\int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta = \left[ \frac{1}{2} \sin^2 \theta \right]_0^{\frac{\pi}{2}} = \frac{1}{2}; \quad \int_0^{\frac{\pi}{2}} \sin \theta d\theta = [-\cos \theta]_0^{\frac{\pi}{2}} = 1.$$

$$\Rightarrow \frac{EIW}{R^3} = H_0^2 \frac{\pi}{4} + V_0^2 \left( \frac{\pi}{2} - 2 + \frac{\pi}{4} \right) - 2H_0 V_0 \left( 1 - \frac{1}{2} \right)$$

$$\Rightarrow \frac{EIW}{R^3} = \frac{P^2}{4\pi} + \frac{P^2}{4} \left( \frac{3\pi}{4} - 2 \right) - \frac{P^2}{\pi} \left( \frac{1}{2} \right) \quad \text{ou} \quad \begin{cases} H_0 = -\frac{P}{\pi} \\ V_0 = -\frac{P}{2} \end{cases}$$

$$\frac{EIW}{R^3} = \frac{P^2}{16\pi} (4 + 3\pi^2 - 8\pi - 8)$$

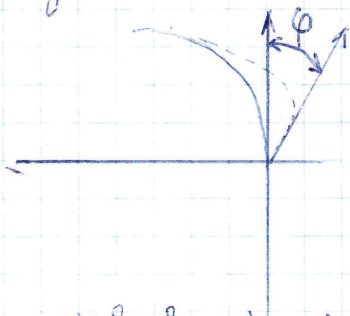
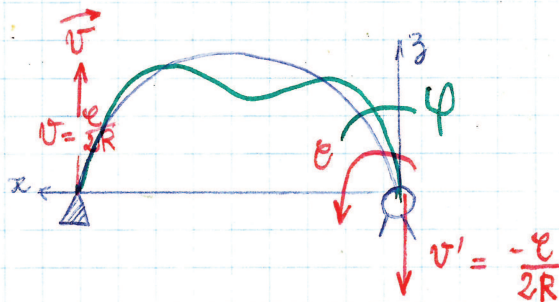
$$\frac{dW}{dP} = \frac{PR^3}{8EI\pi} (3\pi^2 - 8\pi - 4)$$

EX 14 EMK p 10

EX: 14 CHAP 10.

même figure SIA.

en fait.



Calcul de M  $M = HR \sin \theta + \frac{P}{2} R (1 - \cos \theta)$

$H = -\frac{P}{\pi} \Rightarrow M = \frac{PR}{2\pi} (-2 \sin \theta + \pi(1 - \cos \theta))$

Calcul de  $\mathcal{R}$ .

$$\mathcal{R} = - \left[ e - \frac{e}{2R} R (1 - \cos \theta) \right]$$

$$\mathcal{R} = -\frac{e}{2} (2 - (1 + \cos \theta)) = -\frac{e}{2} (1 + \cos \theta)$$



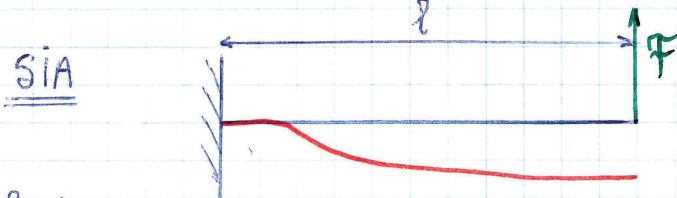
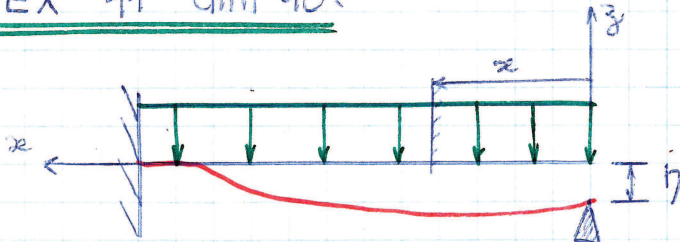
~~ferriculant~~ 
$$\mathcal{E}\varphi = 2 \int_0^{\frac{\pi}{2}} \frac{\mathcal{M}}{EI} R d\theta$$

~~$$\Rightarrow \mathcal{E}\varphi = \frac{R^2 P \mathcal{E}}{EI 2R} \int_0^{\frac{\pi}{2}} [2 \sin\theta + 2 \sin\theta \cos\theta - \pi(1 + \cos^2\theta)] d\theta$$~~

~~$$\Rightarrow \varphi = \frac{PR^2}{2\pi EI} (2 + 1 - \pi(\frac{\pi}{2} + \frac{\pi}{4}))$$~~

~~$$\varphi = \frac{PR^2}{2\pi EI} (3 - \frac{\pi^2}{4})$$~~

EX 11 CHAP 10.



- ferriculant

$$F\eta = \int_L \frac{\mathcal{M}}{EI} ds$$

Calcul de M

$$x \in (0, l) \quad M = -[V_0 x + \frac{P x^2}{2}]$$

Calcul de \mathcal{M}

$$x \in (0, l) \quad \mathcal{M} = -[F x]$$

$$EI \cdot F\eta = F \int_0^l (V_0 x^2 + \frac{P x^3}{2}) dx \quad \forall F$$

$$\Rightarrow EI \eta = \frac{V_0 l^3}{3} + \frac{P l^4}{8} \Rightarrow V_0 = \frac{3EI\eta}{l^3} - \frac{3Pl}{8}$$

Suite. EXO 14 CHAP 10.

$$\theta \in [\frac{\pi}{2}, \pi] \quad M = -VR [1 - (\cos(\pi - \theta))] + HR \sin(\pi - \theta)$$

$$M = \frac{PR}{2} (1 + \cos\theta) - \frac{PR}{\pi} \sin\theta$$

$$\mathcal{E}\varphi = \int_0^{\frac{\pi}{2}} \frac{\mathcal{M}}{EI} ds + \int_{\frac{\pi}{2}}^{\pi} \frac{\mathcal{M}}{EI} ds$$

$$A = \int_{\frac{\pi}{2}}^{\pi} \mathcal{M} ds = -\frac{e}{2} \int_{\frac{\pi}{2}}^{\pi} \left[ \frac{PR}{2} (1 + \cos \theta)^2 - \frac{PR}{\pi} (\sin \theta + \sin \theta \cos \theta) \right] R d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} \cos \theta d\theta = -1.$$

$$\int_{\frac{\pi}{2}}^{\pi} \cos^2 \theta d\theta = \int_{\frac{\pi}{2}}^{\pi} \frac{1 + \cos 2\theta}{2} d\theta = \frac{\pi}{4}$$

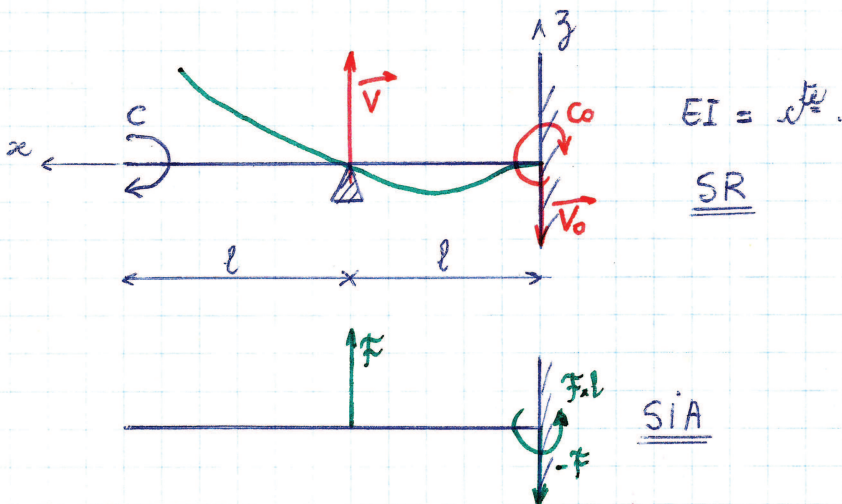
$$\int_{\frac{\pi}{2}}^{\pi} \sin \theta d\theta = +1 \quad \text{et} \quad \int_{\frac{\pi}{2}}^{\pi} \sin \theta \cos \theta = -\frac{1}{2}$$

$$A = -\frac{eR}{2} \left[ \frac{PR}{2} \left( \frac{\pi}{2} - 2 + \frac{\pi}{4} \right) - \frac{PR}{\pi} \left( 1 - \frac{1}{2} \right) \right] = -\frac{e}{2} \frac{PR^2}{2\pi} \left[ \frac{3\pi^2}{4} - 2\pi - 1 \right].$$

$$e\varphi = \frac{ePR^2}{4\pi EI} \left[ \left( 3 - \frac{\pi^2}{4} \right) + \left( 1 + 2\pi - \frac{3\pi^2}{4} \right) \right] \quad \forall e.$$

$$\Rightarrow \varphi = \frac{PR^2}{4\pi EI} (4 + 2\pi - \pi^2)$$

### EX 16 CHAP 10



1) Calcul de  $v$

Calcul de  $M$

$$x \in (0, l) \quad M = -v(l-x) + C$$

$$x \in (l, 2l) \quad M = C$$

Calcul de  $\mathcal{N}$

$$x \in (0, l) \quad \mathcal{N} = -F(l-x)$$

$$x \in (l, 2l) \quad \mathcal{N} = 0$$

th de Fermirolant.

$$F\lambda = \int_L \frac{\mathcal{N}}{EI} ds \quad \text{or} \quad \lambda = 0$$

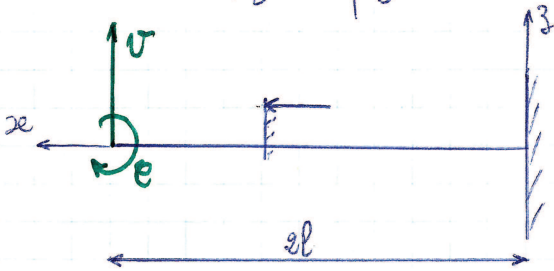
$$0 = FV \int_0^l (l-x)^2 dx - cF \int_0^l (l-x) dx \quad VF$$

$$\Rightarrow V \left[ \frac{(l-x)^3}{-3} \right]_0^l - c \left[ \frac{(l-x)^2}{-2} \right]_0^l = 0$$

$$V \frac{l^3}{3} - c \times \frac{l^2}{2} = 0$$

$$\Rightarrow V = \frac{3c}{2l}$$

Calcul de  $\lambda_B$  et  $\phi_B$ .



Calcul de  $\mathcal{H}$

$$x \in (0, 2l)$$

$$\mathcal{H} = -V(2l-x) + c$$

th de Fermirolant.

$$EI \lambda \mathcal{V} + EI \mathcal{E} \phi = V \int_0^l V(l-x)(2l-x) dx + c \int_0^l -V(l-x) dx + V \int_0^l \dots$$

$$\dots - (2l-x)c + c \int_0^l c dx + \int_l^{2l} -VC(2l-x) dx + \int_l^{2l} cC dx \quad VF, F$$

$$\begin{cases} EI \lambda = V \int_0^l (2l^2 - 3lx + x^2) dx - c \int_0^l (2l-x) dx \\ EI \phi = -V \int_0^l (l-x) dx + c \int_0^l dx \end{cases}$$

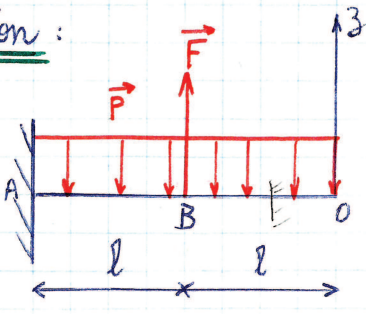
$$\text{Donc } \begin{cases} EI \lambda = Vl^3 \left( 2 - \frac{3}{2} + \frac{1}{3} \right) - 2cl^2 \\ EI \phi = -V \left[ -\frac{(l-x)^2}{2} \right]_0^l + c [x]_0^l = -V \frac{l^2}{2} + 2cl \end{cases}$$

$$\begin{cases} \lambda = \frac{5Vl^3}{6EI} - \frac{2cl^2}{EI} \\ \phi = \frac{2cl}{EI} - \frac{Vl^2}{2EI} \end{cases}$$

on remplace par la valeur de  $V = \frac{3c}{2l}$

Correction :

I)



1) Calcul de  $\lambda_B$  par Castigliano.

$$\lambda_B = \frac{\partial W}{\partial F} \quad \text{avec} \quad W = \int_L \frac{M^2}{2EI} \Rightarrow \lambda_B = \int_L \frac{M}{EI} \frac{\partial M}{\partial F} ds$$

le système est isostatique, le calcul de M ne nécessite pas le calcul des actions de contact.

$$x \in (0, l) \quad M = -p \frac{x^2}{2} \Rightarrow \frac{\partial M}{\partial F} = 0$$

$$x \in (l, 2l) \quad M = -p \frac{x^2}{2} - F(x-l) \Rightarrow \frac{\partial M}{\partial F} = -(x-l)$$

$$\text{Donc} \quad EI \lambda_B = \int_l^{2l} \left[ \frac{p}{2} (x^3 - 2lx^2) + F(x-l)^2 \right] dx = \frac{pl^4}{2} \left( \frac{15}{4} - \frac{7}{3} \right) + \frac{Fl^3}{3}$$

$$\lambda_B = \frac{l^3}{24EI} (17pl + 8F)$$

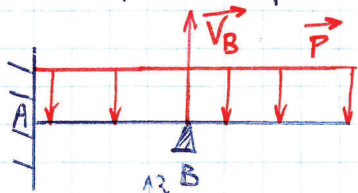
2) Calcul de F si  $\lambda_B = 0$

$$\Rightarrow 17pl + 8F = 0 \Rightarrow F = -\frac{17pl}{8}$$

3) Système mécaniquement correspondant

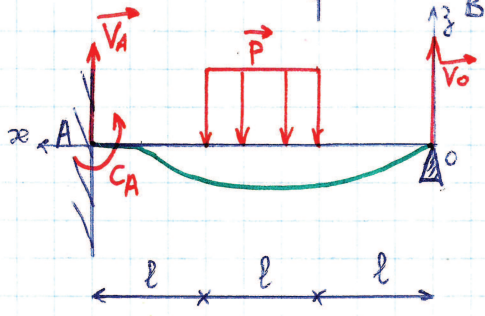
$\lambda_B = 0 \Rightarrow$  appui simple en B

d'où



$$\text{avec} \quad V_B = -\frac{17pl}{8}$$

II)



déformation réelle du SR

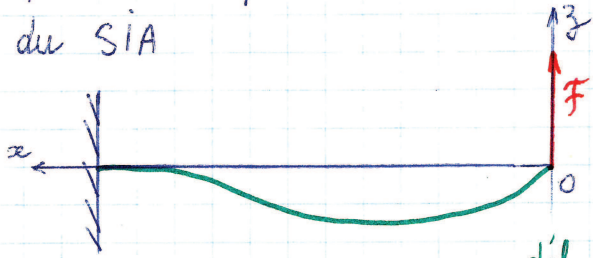
1) Equation de la statique

$$\begin{cases} H_A = 0 & (\text{projection verticale nulle.}) \\ V_A + V_O + pl = 0 \\ 3V_O l + \frac{3pl^2}{2} + C_A = 0 \end{cases}$$

2) Calcul des actions de contact  
 système hyperstatique degré 1.  
 (3 équations 4 inconnues).

équation supplémentaire par Fontvirolant.

Déf du SIA



déf virtuelle SIA  $\equiv$  déf réelle SR.

donc  $\lambda_0 = 0$

$$\sum \mathcal{V} F_{\text{ext}} + \sum \mathcal{V} F_{\text{int}} = 0$$

$$F \lambda_0 - W' = 0 \Rightarrow F \lambda_0 = \int_L \frac{mM}{EI} ds$$

Calcul de  $m$  et  $M$

$$x \in (0, 3l) \quad m = -Fx$$

$$x \in (0, l) \quad M = -V_0 x$$

$$x \in (l, 2l) \quad M = -V_0 x - \frac{P}{2} (x-l)^2$$

$$x \in (2l, 3l) \quad M = -V_0 x - pl \left(x - \frac{3l}{2}\right)$$

Donc on a :

$$0 = \int_L m M ds = \int_0^l V_0 x^2 dx + \int_l^{2l} \left[ V_0 x^2 + \frac{Px}{2} (x-l)^2 \right] dx + \dots$$

$$\dots \int_{2l}^{3l} \left[ V_0 x^2 + pl \left(x^2 - \frac{3lx}{2}\right) \right] dx$$

$$= \int_0^{3l} V_0 x^2 dx + \frac{P}{2} \int_l^{2l} (x^3 - 2lx^2 + l^2 x) dx + pl \int_{2l}^{3l} \left(x^2 - \frac{3lx}{2}\right) dx$$

$$\Rightarrow V_0 \frac{27l^3}{3} + \frac{pl^4}{2} \left(\frac{15}{4} - \frac{14}{3} + \frac{3}{2}\right) + pl^4 \left(\frac{31}{12}\right) = 0$$

$$9V_0 + pl \frac{69}{24} = 0 \Rightarrow V_0 = -\frac{23}{72} pl$$

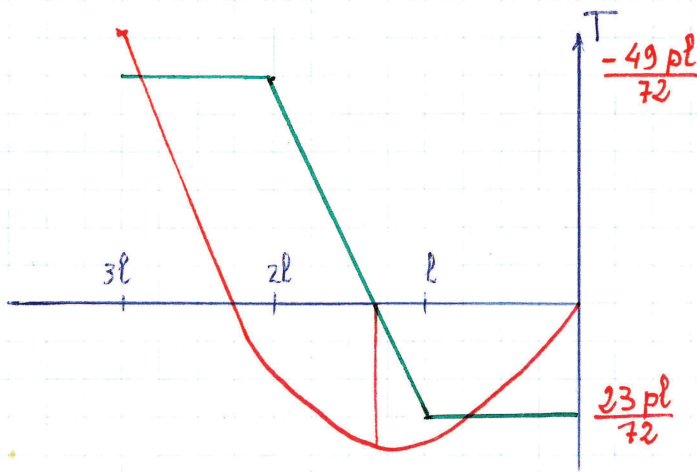
$$V_A = -\frac{49}{72} pl \quad \text{et} \quad C_A = -\frac{13pl^2}{24}$$

$$x \in (0, l) \quad T = -V_0 = \frac{23pl}{72}$$

$$x \in (l, 2l) \quad T = -V_0 - p(x-l) = \frac{23pl}{72} - px + pl$$

$$T = \frac{P}{72} (95l - 72x)$$

$$x \in (2l, 3l) \quad T = V_A = -\frac{49 pl}{72}$$

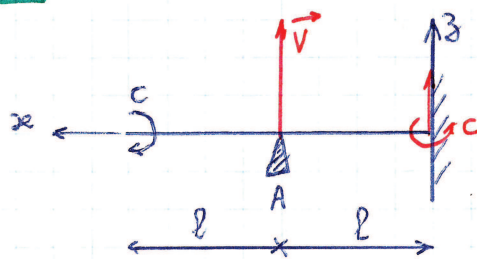


4)  $M_{\text{max}}$  en A

$$M = -\frac{13 pl^2}{24} \quad \sigma = \frac{Mz}{I_y} \quad \text{avec } z = \frac{h}{2} \quad I_y = \frac{bh^3}{12}$$

$$\sigma_{\text{max}} = \frac{-13 pl^2 \times 12 \times h}{24 bh^3 \times 2} = -\frac{13 pl^2}{4 bh^2}$$

### EX 17 CHAP 10.



On sait que  $V = \frac{3C}{2l}$

ligne élastique:

$$\frac{d^2 \lambda}{dx^2} = -\frac{M}{EI}$$

calcul de M

$$x \in (0, l) \quad M = -V(l-x) + C = -\frac{3C}{2l}(l-x) + C = \frac{C}{2l}(3x-l)$$

$$x \in (l, 2l) \quad M = C$$

D'où

$$x \in (0, l) \quad EI \frac{d^2 \lambda}{dx^2} = \frac{C}{2l}(l-3x)$$

$$EI \frac{d\lambda}{dx} = \frac{C}{2l}(lx - \frac{3x^2}{2}) + k_1$$

$$EI \lambda = \frac{C}{2l}(\frac{l x^2}{2} - \frac{3x^3}{6}) + k_1 x + k_2$$

$x \in (l, 2l)$

$$EI \frac{d^2 \lambda}{dx^2} = -C$$

$$EI \frac{d\lambda}{dx} = -Cx + k_3$$

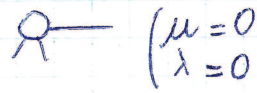
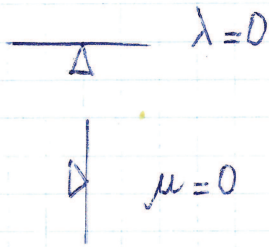
$$EI \lambda = -C \frac{x^2}{2} + k_3 x + k_4$$

Recherche des conditions aux limites.

$$x=0 \begin{cases} \lambda=0 & (1) \\ \frac{d\lambda}{dx}=0 & (2) \end{cases} \quad x=l \begin{cases} \lambda_D=0 & (3) \\ \lambda_G=0 & (4) \\ \left(\frac{d\lambda}{dx}\right)_G = \left(\frac{d\lambda}{dx}\right)_D & (5) \end{cases}$$

$2m+k = 2 \times 2 + 1 = 5$  Conditions limites.

$\left\{ \begin{array}{l} k: \text{degré hyperstatéité} \\ m: \text{intervalle de définition de } M \end{array} \right.$



(1)  $\Rightarrow k_2 = 0$

(2)  $\Rightarrow k_4 = 0$

(5)  $\Rightarrow \frac{3c}{2l} \left( \frac{2l^2}{2} - \frac{3l^2}{2} \right) = -cl + k_3 \Leftrightarrow k_3 = \frac{3cl}{4}$

(3)  $\Rightarrow -\frac{cl^2}{2} + k_3 l + k_4 = 0 \Rightarrow k_4 = -\frac{cl^2}{4}$

$x \in (0, l)$   $EI\lambda = \frac{c}{2l} \left( \frac{l x^2}{2} - \frac{3x^3}{6} \right)$  (A)

$x \in (l, 2l)$   $EI\lambda = -\frac{cx^2}{2} + \frac{3clx}{4} - \frac{cl^2}{4}$  (B)

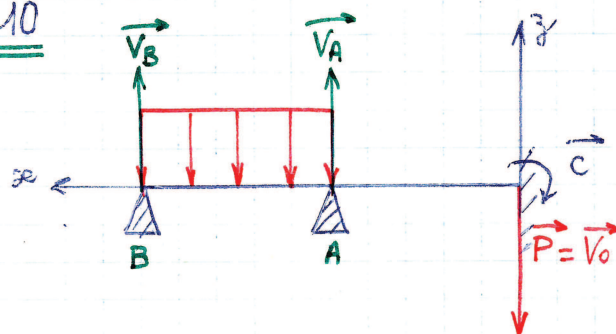
(A)  $\Leftrightarrow EI\lambda = \frac{cx^2}{4l} (l-x)$  pour  $x=0$   $\left\{ \begin{array}{l} \lambda=0 \\ \lambda'=0 \end{array} \right. \Rightarrow x^2$  est racine (double)

pour  $x=l$   $\lambda=0$  donc l racine.

(B)  $\Leftrightarrow EI\lambda = \frac{c}{4} (-2x^2 + 3lx - l^2)$

$\Leftrightarrow EI\lambda = \frac{c}{4} (x-l)(-2x+l)$

EX 19 CHAP 10



on place un encastrement.

## Statique :

$$\begin{cases} V_A + V_B + V_0 + pl = 0 \\ -V_A \cdot l - 2V_B l + C_0 - \frac{3pl^2}{2} = 0 \end{cases}$$

Conditions limites

$$x=0 \begin{cases} \lambda = 0 \quad (1) \\ \frac{d\lambda}{dx} = 0 \quad (2) \end{cases} \quad x=l \begin{cases} \lambda_G = \lambda_D = 0 \quad (3) \quad (4) \\ \left(\frac{d\lambda}{dx}\right)_G = \left(\frac{d\lambda}{dx}\right)_D \quad (5) \end{cases} \quad x=2l \quad \lambda = 0 \quad (6)$$

Calcul de M.

$$x \in (0, l) \quad M = -(C_0 + V_0 \cdot x)$$

$$x \in (l, 2l) \quad M = -(C_0 + V_0 x + V_A(x-l) + p \frac{(x-l)^2}{2}).$$

ligne élastique

$$x \in (0, l) \quad EI \frac{d^2 \lambda}{dx^2} = C_0 + V_0 x$$

$$EI \frac{d\lambda}{dx} = C_0 x + V_0 \frac{x^2}{2} + k_1$$

$$EI \lambda = C_0 \frac{x^2}{2} + V_0 \frac{x^3}{6} + k_1 x + k_2$$

$$x \in (l, 2l) \quad EI \frac{d^2 \lambda}{dx^2} = C_0 + V_0 x + V_A(x-l) + p \frac{(x-l)^2}{2}$$

$$EI \frac{d\lambda}{dx} = C_0 x + V_0 \frac{x^2}{2} + V_A \frac{(x-l)^2}{2} + p \frac{(x-l)^3}{6} + k_3$$

$$EI \lambda = C_0 \frac{x^2}{2} + V_0 \frac{x^3}{6} + V_A \frac{(x-l)^3}{6} + p \frac{(x-l)^4}{24} + k_3 x + k_4$$

$$\left. \begin{array}{l} (1) \Rightarrow k_2 = 0 \\ (2) \Rightarrow k_1 = 0 \\ (3) \Rightarrow k_3 l + k_4 = 0 \\ (5) \Rightarrow k_3 = 0 \end{array} \right\} k_1 = k_2 = k_3 = k_4 = 0$$

$$(4) \Rightarrow C_0 + V_0 \frac{l}{3} = 0 \quad \Rightarrow \quad C_0 = -\frac{V_0 l}{3}$$

$$2C_0 l^2 + 4 \frac{V_0 l^3}{3} + V_A \frac{l^3}{6} + \frac{pl^4}{24} = 0$$

$$\Leftrightarrow 2C_0 + 4 \frac{V_0 l}{3} + V_A \frac{l}{6} + \frac{pl^2}{24} = 0$$

$$V_A + V_B + V_0 + pl = 0$$



$$-V_A l - 2V_B l + C_0 - \frac{3}{2} pl^2 = 0$$

$$\frac{2}{3} V_0 + V_A \frac{1}{6} + \frac{pl}{24} = 0$$

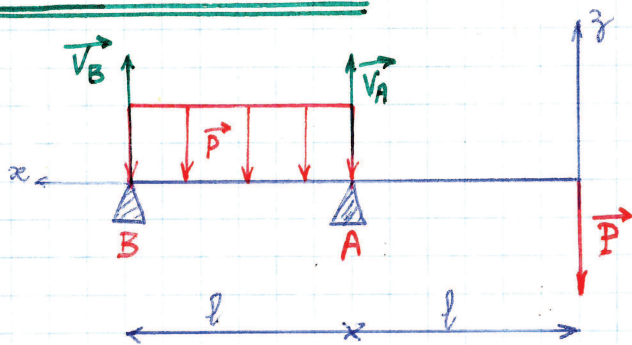
$$V_B = -V_A - V_0 - pl$$

$$V_A + \frac{5}{3} V_0 + \frac{pl}{2} = 0$$

$$\Rightarrow \begin{cases} 16V_0 + 4V_A + pl = 0 \\ 10V_0 + 6V_A + 3pl = 0 \end{cases} \Rightarrow \begin{cases} V_0 = \frac{3}{28} pl \\ V_A = -\frac{19}{28} pl \end{cases}$$

$$\begin{cases} V_B = -\frac{3}{7} pl \\ C_0 = -\frac{pl^2}{28} \end{cases}$$

EX 19 + 20 CHAP 10.



Calcul actions de contact.

$$\begin{cases} V_A + V_B + pl + P = 0 \\ \frac{pl^2}{2} + V_A l + 2Pl = 0 \end{cases}$$

$$V_A = -\frac{3}{2} pl \quad (\text{ou } P = \frac{pl}{2})$$

$$\Rightarrow V_B = 0$$

ligne élastique :  $\frac{d^2 \lambda}{dx^2} = \frac{-M}{EI}$

Calcul de M

$$x \in (0, l) \quad M = -Px$$

$$x \in (l, 2l) \quad M = -\left[ Px + p \frac{(x-l)^2}{2} + V_A (x-l) \right]$$

Conditions limites  $2x$  (intervalle définition de M) + d'hyperstativité!

$$x = l \quad \begin{cases} \lambda_D = \lambda_G = 0 & (1) (2) \\ \left( \frac{d\lambda}{dx} \right)_G = \left( \frac{d\lambda}{dx} \right)_D & (3) \end{cases} \quad x = 2l \quad \lambda = 0 \quad (4)$$

$$x \in (0, l)$$

$$EI \frac{d^2 \lambda}{dx^2} = \frac{pl}{2} x \Rightarrow EI \frac{d\lambda}{dx} = \frac{pl}{2} \left( \frac{x^2}{2} + k_1 \right)$$

$$\Leftrightarrow EI \lambda = \frac{pl}{2} \left( \frac{x^3}{6} + k_1 x + k_2 \right)$$

$$x \in (l, 2l)$$

$$EI \frac{d^2 \lambda}{dx^2} = \frac{1l}{2} x + \frac{1}{2} (x-l)^2 - \frac{3l}{2} (x-l)$$

$$EI \frac{d\lambda}{dx} = \frac{1l}{4} x^2 + \frac{1}{6} (x-l)^3 - \frac{3}{4} l (x-l)^2 + k_3$$

$$EI \lambda = \frac{1l}{12} x^3 + \frac{1}{24} (x-l)^4 - \frac{3l}{12} (x-l)^3 + k_3 x + k_4$$

on les CL

$$\left\{ \begin{aligned} 0 &= \frac{1l^4}{12} + k_3 l + k_4 \\ 1l \left( \frac{l^3}{6} + k_1 l + k_2 \right) &= 0 \\ \frac{1l^3}{4} + \frac{1l k_1}{2} &= \frac{1l^3}{4} + k_3 \\ 1l^4 \left( \frac{3}{3} + \frac{1}{24} - \frac{3}{12} \right) + 2k_3 l + k_4 &= 0 \end{aligned} \right.$$

$$\Leftrightarrow \begin{cases} k_3 l + k_4 = -\frac{1l^4}{12} \\ 2k_3 l + k_4 = -\frac{11}{24} 1l^4 \end{cases} \Rightarrow \begin{cases} k_3 = -\frac{9}{24} 1l^3 \\ k_4 = \frac{7}{24} 1l^4 \end{cases}$$

$$x \in (0, l) \Rightarrow k_1 = -\frac{3}{4} l^2 \text{ et } k_2 = \frac{7l^3}{12}$$

$$EI \lambda = \frac{1l}{2} \left( \frac{x^3}{6} + k_1 x + k_2 \right) = \frac{1l}{2} \left( \frac{x^3}{6} - \frac{3}{4} l^2 x + \frac{7}{12} l^3 \right)$$

$$EI \lambda = \frac{1l}{24} (x-l) (2x^2 + 2lx - 7l^2)$$

( $x=l$  est racine car  $\lambda=0$ )

$$\begin{array}{r|l} 2x^3 + 0x^2 - 9l^2 x + 7l^3 & x-l \\ 0 \quad 2lx^2 - 9l^2 x + 7l^3 & \\ 0 \quad -7l^2 x + 7l^3 & \end{array} \quad \begin{array}{l} 2x^2 + 2lx - 7l^2 \end{array}$$

$$x \in (l, 2l)$$

$$EI \lambda = \frac{1l}{24} \left[ 2lx^3 + (x-l)^4 - 6l(x-l)^3 - 9l^3 x + 7l^4 \right]$$

$$EI \lambda = \frac{1l}{24} \left[ (x-l) \left[ (x-l)^3 - 6l(x-l)^2 + (2lx^2 + 2lx - 7l^3) \right] \right]$$

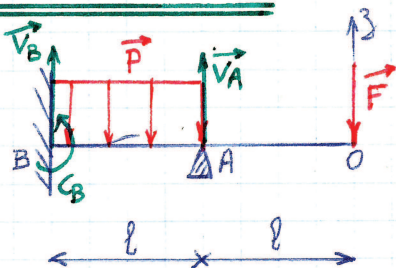
$$\begin{array}{r|l}
 2lx^3 + 0x^2 - 9l^3x + 7l^4 & x-l \\
 \hline
 2lx^2 - 9l^3x + 7l^4 & 2lx^2 + 2l^2x - 7l^3 \\
 -7l^3x + 7l^4 & 
 \end{array}$$

On obtient :

$$\begin{array}{r|l}
 x^3 - 3xl^2 + 3xl^2 - l^3 & \\
 -6x^2l + 12xl^2 - 6l^3 & \\
 2x^2l + 2xl^2 - 7l^3 & \\
 \hline
 x^3 - 7x^2l + 17xl^2 - 14l^3 & x-2l \\
 -5x^2l + 17xl^2 - 14l^3 & x^2 - 5xl + 7l^2 \\
 7xl^3 - 14l^3 & \\
 0 & 
 \end{array}$$

Donc  $EI\lambda = \frac{P}{24} (x-l)(x-2l)(x^2 - 5xl + 7l^2)$ .

### EX 21 CHAP 10



ligne élastique

$$\frac{d^2\lambda}{dx^2} = \frac{-M}{EI}$$

Calcul de M

$$x \in (0, l) \quad M = -F \cdot x$$

$$x \in (l, 2l) \quad M = -\left[ Fx + V_A(x-l) + \frac{P}{2}(x-l)^2 \right]$$

Equations

$$x \in (0, l) \quad EI \frac{d^2\lambda}{dx^2} = Fx$$

$$EI \frac{d\lambda}{dx} = F \frac{x^2}{2} + k_1$$

$$EI \lambda = F \frac{x^3}{6} + k_1 x + k_2$$

$$x \in (l, 2l) \quad EI \frac{d^2 \lambda}{dx^2} = Fx + V_A (x-l) + \frac{P}{2} (x-l)^2$$

$$EI \frac{d\lambda}{dx} = F \frac{x^2}{2} + V_A \frac{(x-l)^2}{2} + \frac{P}{2} \frac{(x-l)^3}{3} + k_3$$

$$EI \lambda = F \frac{x^3}{6} + V_A \frac{(x-l)^3}{6} + \frac{P}{2} \frac{(x-l)^4}{12} + k_3 x + k_4$$

$\exists 2m+k$  conditions limite = 5 (ici)

$$x=l \quad \begin{cases} \lambda_G = \lambda_D = 0 \\ \left(\frac{d\lambda}{dx}\right)_G = \left(\frac{d\lambda}{dx}\right)_D \end{cases} \quad x=2l \quad \begin{cases} \lambda = 0 \\ \frac{d\lambda}{dx} = 0 \end{cases}$$

$$k_1 l + k_2 = k_3 l + k_4 \quad \rightarrow \quad k_2 = k_4$$

$$\textcircled{2} \quad \frac{Fl^3}{6} + k_1 l + k_2 = 0$$

$$k_1 = k_3$$

$$1 \quad \frac{8Fl^3}{6} + V_A \frac{l^3}{6} + \frac{Pl^4}{24} + 2k_3 l + k_4 = 0$$

$$-2l \quad 2Fl^2 + V_A \frac{l^2}{2} + \frac{Pl^3}{6} + k_3 = 0 \quad \textcircled{1} \quad (\text{on multiplie par } -2l)$$

$$-\frac{8}{3} Fl^3 - \frac{5}{6} V_A l^3 - \frac{7}{24} Pl^4 + k_4 = 0 \quad \textcircled{3}$$

$$\textcircled{1} \Rightarrow k_3 = -2Fl^2 - V_A \frac{l^2}{2} - \frac{Pl^3}{6} = k_4 \quad \text{qu'on reporte dans } \textcircled{2}$$

on remplace  $k_2$  par  $k_4$  ( $k_2 = k_4$ ).

$$-\frac{11}{6} Fl^3 - V_A \frac{l^3}{2} - \frac{Pl^4}{6} + k_4 = 0 \quad (\text{on multiplie par } (-1) \text{ et}$$

on ajoute à  $\textcircled{3}$ )

$$\Rightarrow -\frac{5}{6} Fl^3 - \frac{V_A l^3}{3} - \frac{Pl^4}{8} = 0$$

$$\Rightarrow V_A = -\frac{5}{2} F - \frac{3}{8} Pl$$

$$\text{D'où} \quad k_4 = \frac{7Fl^3}{12} - \frac{Pl^4}{48} = k_2$$

$$k_3 = -\frac{3}{4} Fl^2 + \frac{Pl^3}{48} = k_1$$

Donc on trouve:

$$EI\lambda = \frac{Fx^3}{6} + \left(-\frac{3Fl^2}{4} + \frac{Pl^3}{48}\right)x + \frac{7Fl^3}{12} - \frac{Pl^4}{48} \quad (x \in (0, l))$$

$$EI\lambda = \frac{Fx^3}{6} + \left(-\frac{5F}{12} - \frac{3pl}{48}\right)(x-l)^3 + \frac{p}{24}(x-l)^4 + \left(-\frac{3Fl^2}{4} + \frac{Pl^3}{48}\right)x + \frac{7}{12}Fl^3 \dots$$

$$\dots - \frac{pl^4}{48}$$

travail minimum au moment de flexion

$$x \in (0, l) \quad M = -Fx$$

$$x \in (l, 2l) \quad M = -Fx - p\frac{(x-l)^2}{2} + \left(\frac{5F}{2} + \frac{3pl}{8}\right)(x-l)$$

Calcul des valeurs particulières de M

$$x = l \quad M_1 = -Fl$$

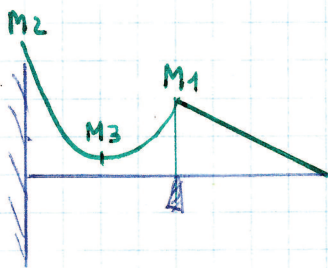
$$x = 2l \quad M_2 = -2Fl - \frac{pl^2}{2} + \left(\frac{5F}{2} + \frac{3pl}{8}\right)l$$

$$M = \frac{Fl}{2} - \frac{pl^2}{8}$$

$$\frac{dM}{dx} = -F - p(x-l) + \frac{5F}{2} + \frac{3pl}{8} = \frac{3}{2}F - px + \frac{11}{8}pl = 0$$

$$\Rightarrow x = \frac{3}{2}\frac{F}{p} + \frac{11}{8}l$$

$$M = -\frac{3}{2}\frac{F^2}{p} - \frac{11}{8}Fl - \frac{p}{2}\left(\frac{3}{2}\frac{F}{p} - \frac{3}{8}l\right)^2 + \left(\frac{5}{2}F + \frac{3pl}{8}\right)\left(\frac{3}{2}\frac{F}{p} - \frac{3}{8}l\right)$$



$$1) \quad M_2 > 0 \quad \frac{Fl}{2} - \frac{pl^2}{8} \geq 0 \Rightarrow F \geq \frac{pl}{4}$$

$M_1 = M_2$  donne la condition

$$2) \quad M_2 < 0 \Rightarrow F < \frac{pl}{4} \quad M_1 = |M_2| \text{ d'où la condition.}$$

Il y aurait 2 autres cas identiques en comparant  $M_1/M_3$  et  $M_2/M_3$

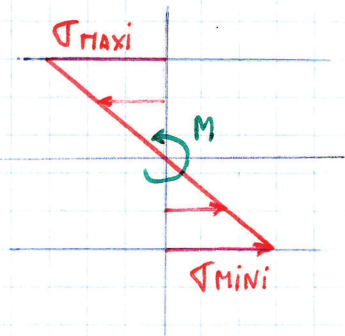
On a

$$M_{\text{maxi}} = \frac{-15 Pa}{27}$$

$$\sigma_{\text{MAXI}} = \frac{M_{\text{maxi}} z_{\text{maxi}}}{I_y} \leq R_e$$

$$\Leftrightarrow \frac{-15 Pa \times r \times 4}{27 \pi r^4} \leq R_e$$

$$\Leftrightarrow r \geq \sqrt[3]{\frac{-20 Pa}{9 \pi R_e}}$$



il faut  $\sigma_{\text{MAXI}} < R_e$   
et  $\sigma_{\text{MINI}} < R_c$

si  $R_e$  et  $R_c$  sont différents il faut prendre le  $R$  pour lequel  $|\sigma|$  est maxi.

AN

$$P = 50 \text{ daN}$$

$$R_e = 10 \text{ daN/mm}^2$$

$$a = \frac{1}{3} \text{ m}$$

$$r \geq 10,86 \text{ mm}$$

La contrainte  $\tau$  est-elle supportée par cet axe.

$$T_{\text{MAXI}} = \frac{-23 P}{27} \Rightarrow \tau_{\text{MAXI}} = \frac{T_{\text{MAXI}}}{S_z}$$

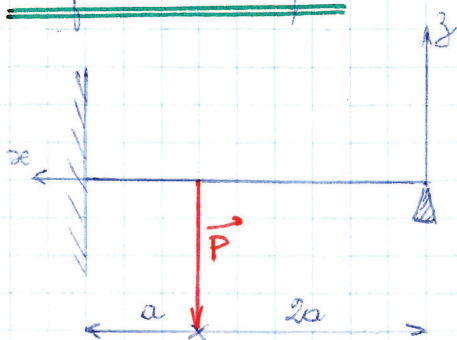
$$\text{et } \tau_{\text{MAXI}} \leq R_{qz}$$

$$\Leftrightarrow \frac{T_{\text{MAXI}}}{S_z} \leq R_{qz} \Leftrightarrow \frac{-4 \times 23 P}{3 \times 5 \times 27} \leq R_{qz}$$

$$\Leftrightarrow \frac{-32 P \times 4}{81 \pi D^2} \leq R_{qz} \Leftrightarrow \tau_{\text{MAXI}} = 0,17 \text{ daN/mm}^2$$

La contrainte tangentielle est très largement supportée.

ligne élastique



$$\left\{ \begin{array}{l} x \in (0, 2a) \\ M = \frac{4Px}{27} \end{array} \right.$$

$$\left\{ \begin{array}{l} x \in (2a, 3a) \\ M = \frac{P}{27} (54a - 23x) \end{array} \right.$$

$$\underline{CL} \quad 2m + k = 5 \cdot CL.$$

$$x=0 \quad \lambda = 0 \quad 1$$

$$x = 2a \quad \left(\frac{d\lambda}{dx}\right)_0 = \left(\frac{d\lambda}{dx}\right)_G \quad \text{et} \quad \lambda_0 = \lambda_G$$

$$x = 3a \quad \lambda = 0 \quad \frac{d\lambda}{dx} = 0$$

$x \in (0, 2a)$

$$\frac{d^2\lambda}{dx^2} = \frac{-M}{EI} \rightarrow \frac{27EI}{P} \frac{d^2\lambda}{dx^2} = -4x$$

$$\Leftrightarrow \frac{27EI}{P} \frac{d\lambda}{dx} = -2x^2 + k_1$$

$$\frac{27EI}{P} \lambda = -\frac{2x^3}{3} + k_1x + k_2$$

$x \in (2a, 3a)$

$$\frac{27EI}{P} \frac{d^2\lambda}{dx^2} = 23x - 54a$$

$$\frac{27EI}{P} \frac{d\lambda}{dx} = \frac{23x^2}{2} - 54ax + k_3$$

$$\frac{27EI}{P} \lambda = \frac{23x^3}{6} - 27ax^2 + k_3x + k_4.$$

$$(1) \Rightarrow k_2 = 0$$

$$(4) \Rightarrow 0 = \frac{23 \times 9a^3}{2} - 27 \times 9a^3 + 3a k_3 + k_4$$

$$(5) \Rightarrow 0 = \frac{23 \times 9a^2}{2} - 3 \times 54a^2 + k_3$$

$$(2) \Rightarrow 46a^2 - 108a^2 + k_3 = -8a^2 + k_4$$

$$\begin{cases} k_3 = a^2 \left( 3 \times 54 - 23 \times \frac{9}{2} \right) = \frac{117}{2} a^2 & \text{et } k_2 = 0 \\ k_4 = a^2 \left( 46 - 108 + 8 + \frac{117}{2} \right) = \frac{9}{2} a^2 \\ k_4 = a^3 \left( 27 \times 9 - \frac{23 \times 9}{2} - \frac{3 \times 117}{2} \right) = -36 a^3. \end{cases}$$

$\Rightarrow x \in (0, 2a)$

$$\lambda = \frac{P}{27EI} \left[ -\frac{2x^3}{3} + \frac{9}{2} a^2 x \right]$$

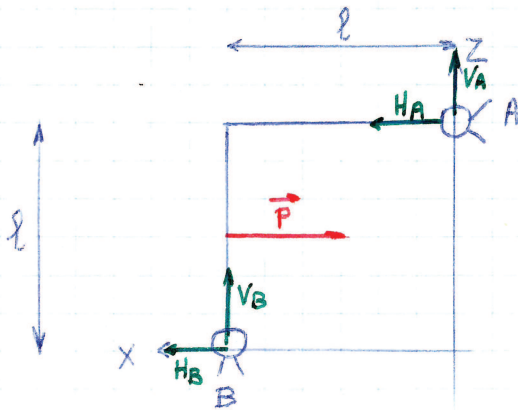
$$x \in (2a, 3a)$$

$$\lambda = \frac{P}{27EI} \left[ \frac{23x^3}{6} - 27ax^2 + \frac{117}{3} a^2 x - 36a^3 \right]$$

$$x = 2a \Rightarrow \lambda = \left( \frac{-2 \times 8a^3}{3} + \frac{9}{2} a^2 \times 2a \right) = \frac{11Pa^3}{81EI}$$

$$\lambda \text{ à l'alamb de } \tau = \frac{11Pa^3}{81EI}$$

EX 26 CHAP 10.



statique:

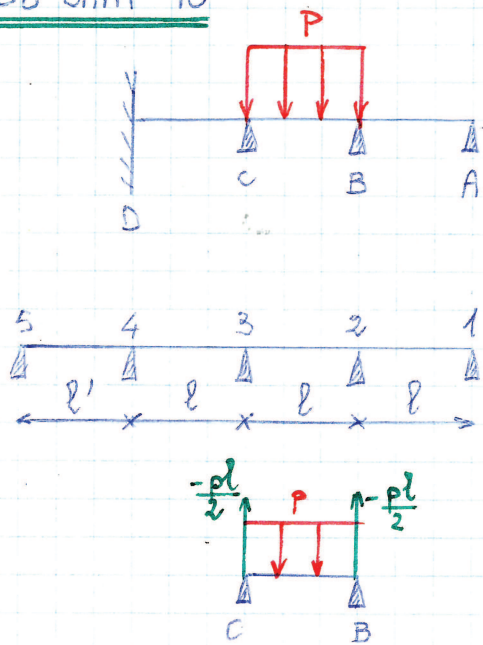
$$\begin{cases} H_A + H_B + P = 0 \\ V_A + V_B = 0 \quad (\text{proj } z) \\ -P \frac{l}{2} - H_B l - V_B \cdot l = 0 \end{cases}$$

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+ 38 CHAP 10



EX 38 CHAP 10



moment de flexion dans les travées  
(≠ (BC)) =  $\mu = 0$

Calcul des moments sur appuis

TMM valeurs particulières  $M_1 = M_5 = 0$

soit  $i = 2$   $l M_3 + 4l M_2 = -\frac{6}{l} \int_0^l \frac{Pxe}{2} (l-x)^2 dx = -\frac{pl^3}{4}$   
 $i = 3$   $l M_4 + 4l M_3 + l M_2 = -\frac{6}{l} \int_0^l \frac{Pxe^2}{2} (l-x) dx = -\frac{pl^3}{4}$   
 $i = 4$   $l M_4 + M_3 = 0 \Rightarrow -\frac{M_3}{2} = M_4$

$M_3 + 4 M_2 = -\frac{pl^2}{4} \Rightarrow M_2 = -\frac{pl^2}{16} - \frac{M_3}{4}$

$M_4 + 4 M_3 + M_2 = -\frac{pl^2}{4}$

$-M_3 \left( \frac{1}{2} - 4 + \frac{1}{4} \right) = -\frac{pl^2}{4} + \frac{pl^2}{16} \Rightarrow M_3 = -\frac{3pl^2}{52}$

$\Rightarrow M_4 = \frac{3pl^2}{104}$  et  $M_2 = -\frac{pl^2}{16} + \frac{3pl^2}{208} = -\frac{5pl^2}{104}$

Expressions de M

$x \in (0, l)$  on sait que  $M - \mu = M_{i-1} + \frac{M_i - M_{i-1}}{l_{i-1}} x$

$\Rightarrow M = -\frac{5pl^2 x}{104l} = -\frac{5plx}{104}$  et  $T = -\frac{5pl}{104} = -V_A = -(F_{extD})$

$x \in (l, 2l)$   $\left( -\frac{3pl^2}{52} + \frac{5pl^2}{104} \right) \frac{x}{l} - \frac{5pl^2}{104} = M - \mu$

$M - \mu = \frac{51plx}{104} - \frac{px^2}{2} - \frac{5pl^2}{104} \Rightarrow \frac{51pl}{104} - px = T$

$$x \in (2l, 3l)$$

$$M = \left( \frac{3pl^2}{104} + \frac{3pl^2}{52} \right) \frac{x}{l} - \frac{3pl^2}{52}$$

$$M = \frac{9plx}{104} - \frac{3pl^2}{52} \Rightarrow T = \frac{9pl}{104}$$

### Calcul des réactions

$$V_A = \frac{5pl}{104}$$

$$V_B = T_D - T_G = \frac{-5pl}{104} - \frac{51pl}{104} = \frac{-56pl}{104}$$

↑  
pour  $x=0$  (réfère flottant).

$$V_C = \frac{-53pl}{104} - \frac{9pl}{104} = \frac{-31pl}{E_d}$$

↑  
pour  $x=l$

$$V_D = \frac{9pl}{104} \quad \text{et} \quad e_D = \frac{3pl^2}{104}$$

### ligne élastique

$$\frac{d^2\lambda}{dx^2} = \frac{-M}{EI}$$

$$x \in (0, l) \quad EI \frac{d^2\lambda}{dx^2} = \frac{5plx}{104} \Rightarrow EI \frac{d\lambda}{dx} = \frac{5plx^2}{208} + k_1$$

$$\Rightarrow EI \lambda = \frac{5plx^3}{604} + k_1x + k_2$$

$$x \in (l, 2l) \quad EI \frac{d^2\lambda}{dx^2} = \frac{px^2}{2} - \frac{51plx}{104} + \frac{5pl^2}{104}$$

$$EI \frac{d\lambda}{dx} = \frac{px^3}{6} - \frac{51plx^2}{208} + \frac{5pl^2x}{104} + k_3$$

$$EI \lambda = \frac{px^4}{24} - \frac{51plx^3}{624} + \frac{5pl^2x^2}{208} + k_3x + k_4$$

$$x \in (2l, 3l) \quad EI \frac{d^2\lambda}{dx^2} = \frac{-9plx}{104} + \frac{3pl^2}{52}$$

$$EI \frac{d\lambda}{dx} = \frac{-9plx^2}{208} + \frac{3pl^2x}{52} + k_5$$

$$EI \lambda = \frac{-9plx^3}{624} + \frac{3pl^2x^2}{104} + k_5x + k_6$$

Conditions aux limites  $2n + k = 9$

$$x = 0 \quad \lambda = 0 \quad \text{en A} \\ (x=0) \quad (x=0) \quad \underline{\text{en A}}$$

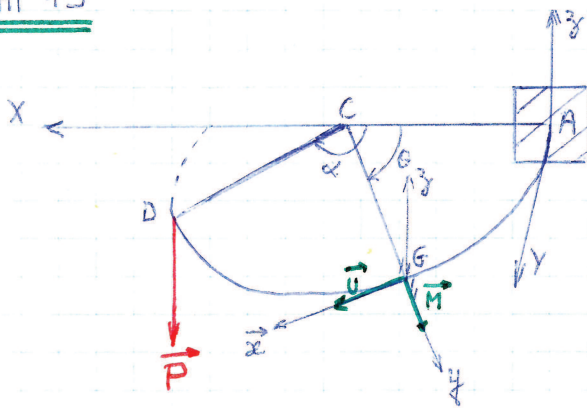
$$\lambda_D = \lambda_G = 0 \quad \left(\frac{d\lambda}{dx}\right)_D = \left(\frac{d\lambda}{dx}\right)_G \quad \text{en B} \\ (x=l) \quad (x=0) \quad (x=l) \quad (x=0)$$

$$\lambda_D = \lambda_G = 0 \quad \left(\frac{d\lambda}{dx}\right)_D = \left(\frac{d\lambda}{dx}\right)_G \quad \text{en C} \\ (x=l) \quad (x=0) \quad (x=l) \quad (x=0)$$

$$\lambda = 0 \quad \left(\frac{d\lambda}{dx}\right) = 0 \quad \text{en D} \\ (x=l) \quad (x=l)$$

faire attention : repère mobile !!!

### EX 1 CHAP 13



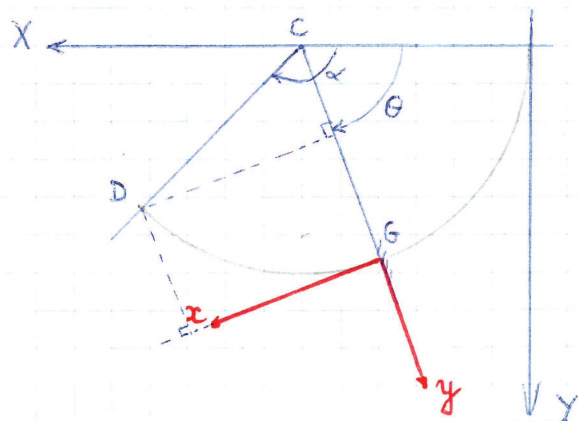
$\vec{x}$  : normale  $\perp$  à la  
facette ( $>0$  sortant à la  
facette)  
 $\vec{y}$  :  $\perp$  à M

$$\{ \text{Forces sup} \} \sim \{ dF \}$$

$$\text{en G} \quad \begin{cases} \vec{R} = \vec{P} \longrightarrow \begin{Bmatrix} N \\ T \end{Bmatrix} \\ \vec{C} = \vec{GD} \wedge \vec{P} \longrightarrow \begin{Bmatrix} U \\ M \end{Bmatrix} \end{cases}$$

$$\vec{GD} = \vec{GC} + \vec{CD} \\ = \begin{pmatrix} 0 \\ -a \\ 0 \end{pmatrix} + \begin{pmatrix} a \sin(\alpha - \theta) \\ a \cos(\alpha - \theta) \\ 0 \end{pmatrix}$$

$$\vec{P} = \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix}$$



$$\vec{GD} \wedge \vec{P} = \begin{pmatrix} Pa [\cos(\alpha - \theta) - 1] \\ -Pa \sin(\alpha - \theta) \\ 0 \end{pmatrix}$$

$\vec{U}$  porté par axe des  $x$  et  $M$  dans le plan de la section ( $M_y, M_z$ ) or ici  $M_z = 0$ ;  $N$  porté par  $x$ ,  $T$  par  $y$  et  $z$ .

$$\Rightarrow \begin{cases} U = Pa [\cos(\alpha - \theta) - 1] \\ M = -Pa \sin(\alpha - \theta) \\ N = 0 \quad \text{ou} \quad \vec{P} \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix} \begin{matrix} \rightarrow N \\ \rightarrow T \end{matrix} \quad \text{ici} \quad T_y = 0 \\ T = P. \end{cases}$$

Calcul de  $W_M, W_U, W_T$ .

$$W_M = \frac{1}{2} \int_0^\alpha \frac{M^2}{EI} ds \quad \text{or} \quad ds = a d\theta$$

$$W_M = \frac{Pa^3}{2EI} \int_0^\alpha \sin^2(\alpha - \theta) d\theta$$

$$\parallel$$

$$\frac{1 - \cos 2(\alpha - \theta)}{2}$$

$$W_M = \frac{Pa^3}{2EI} \cdot \frac{1}{2} \left[ \theta + \frac{\sin 2(\alpha - \theta)}{2} \right]_0^\alpha = \frac{Pa^3}{4EI} \left( \alpha - \frac{\sin 2\alpha}{2} \right)$$

$$W_U = \frac{1}{2} \int_0^\alpha \frac{U^2}{GJ_1} ds \quad \text{avec} \quad G = \frac{2}{5} E \quad \text{et} \quad J_1 = 2I_y$$

$$W_U = \frac{5Pa^3}{8EI} \int_0^\alpha \underbrace{[\cos(\alpha - \theta) - 1]^2}_{\cos(\theta - \alpha)} d\theta = \frac{5Pa^3}{8EI} \int_0^\alpha (\cos^2(\theta - \alpha) - 2\cos(\theta - \alpha) + 1) d\theta$$

$$W_U = \frac{5Pa^3}{8EI} \int_0^\alpha \left[ \frac{1 + \cos 2(\theta - \alpha)}{2} - 2\cos(\theta - \alpha) + 1 \right] d\theta$$

$$W_U = \frac{5Pa^3}{8EI} \left[ \frac{\alpha}{2} + \frac{\sin 2\alpha}{4} - 2\sin \alpha + \alpha \right]$$

$$W_U = \frac{5Pa^3}{8EI} \left[ \frac{3}{2}\alpha + \frac{\sin 2\alpha}{4} - 2\sin \alpha \right]$$

$$W_T = \frac{1}{2} \int_L \frac{T^2}{GS^2} ds \quad \text{avec} \quad \begin{cases} S'_z = \frac{9}{10} S \\ ds = a d\theta \end{cases}$$

$$W_T = \frac{1}{2} \frac{50Pa^2}{18ES} \int_0^\alpha d\theta = \frac{50Pa^2}{18ES} \cdot \frac{1}{2} = \frac{25Pa^2}{18ES}$$

$$W_U = \frac{25Pa^2}{18ES} ; \quad W_U = \frac{5Pa^3}{8EI} \left[ \frac{3}{2} \alpha + \frac{\sin 2\alpha}{4} - 2 \sin \alpha \right]$$

$$W_M = \left( \alpha - \frac{\sin 2\alpha}{2} \right) \frac{Pa^3}{4EI}$$

Comparons  $W_U, W_M, W_T$  pour  $\alpha = \frac{\pi}{2}$

$$W_T = \frac{25Pa^2}{36Ez^2}$$

$$W_U = \frac{5Pa^3}{8E\pi z^4} \left( \frac{3\pi}{4} - 2 \right) = \frac{5Pa^3}{8E\pi z^4} (3\pi - 8) = 0,28 \frac{Pa^3}{Ez^4}$$

$$W_M = \frac{Pa^3}{4E\pi z^4} \left( \frac{\pi}{2} \right) = 0,5 \frac{Pa^3}{Ez^4}$$

On obtient  $W \approx W_M + W_U \approx 0,78 \frac{Pa^3}{Ez^4}$

Calcul de  $\lambda_D$ :

$$W \approx W_U + W_M = \frac{Pa^3}{8EI} \left( \frac{19\alpha}{2} + \frac{1}{4} \sin 2\alpha - 10 \sin \alpha \right)$$

$$\lambda_D = \frac{JW}{JP} = \frac{Pa^3}{4EI} \left( \frac{19\alpha}{2} + \frac{1}{4} \sin 2\alpha - 10 \sin \alpha \right)$$

Pour  $\alpha = \frac{\pi}{2} \Rightarrow \lambda_D = \frac{Pa^3}{4EI} \left( \frac{19\pi}{4} - 10 \right)$

Soit  $\alpha = \pi \quad r' = 0,8 r$

↑ résistance pratique au cisaillement.      ↑ résistance pratique à l'extension

$$M_i = \frac{1}{2} (|M| + \sqrt{M^2 + U^2}) \quad \text{et} \quad U_i = \sqrt{M^2 + U^2};$$

$$M = -Pa \sin(\pi - \theta) = -Pa \sin \theta$$

$$U = Pa(-\cos \theta - 1) = -Pa(1 + \cos \theta)$$

$$\sqrt{M^2 + U^2} = 2Pa \sin \frac{\alpha}{2}$$

$$M_i = \frac{Pa}{2} (-\sin \alpha + 2 \sin \frac{\alpha}{2})$$

pour  $\alpha = \pi$

$$\begin{cases} U = -Pa [\cos \theta + 1] \\ M = -Pa \sin \theta \end{cases}$$

$$U_i = \left( [-Pa (\cos \theta + 1)]^2 + [-Pa \sin \theta]^2 \right)^{\frac{1}{2}}$$

$$U_i = \left[ Pa^2 (\cos^2 \theta + 2 \cos \theta + 1) + Pa^2 \sin^2 \theta \right]^{\frac{1}{2}} = -Pa \left[ 2 (\cos \theta + 1) \right]^{\frac{1}{2}}$$

$$M_i = \frac{1}{2} \left( -Pa \sin \theta + Pa \left[ 2 (\cos \theta + 1) \right]^{\frac{1}{2}} \right) \quad \downarrow \text{ou } P < 0$$

$$\text{or } \begin{cases} \cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2} \\ \sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2} \end{cases}$$

$$\Rightarrow M_i = -\frac{Pa}{2} \left[ +\sin \theta + 2 \left( \cos^2 \frac{\theta}{2} \right)^{\frac{1}{2}} \right] = -\frac{Pa}{2} \left[ +\sin \theta + 2 \cos \frac{\theta}{2} \right]$$

$$M_i = -\frac{Pa}{2} \left( 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} + 2 \cos \frac{\theta}{2} \right) = -Pa \cos \frac{\theta}{2} (\sin \frac{\theta}{2} + 1)$$

$U_i$  maxi pour  $\theta = 0$ .

$$U_i \text{ maxi} = -2Pa$$

$$M_i \text{ maxi } \frac{dM_i}{d\theta} = -\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} - \sin \frac{\theta}{2} = 0$$

$$= \cos \theta - \sin \frac{\theta}{2} = \cos \theta - \cos \left( \frac{\pi}{2} - \frac{\theta}{2} \right) = 0$$

$$\cos \theta = \cos \left( \frac{\pi}{2} - \frac{\theta}{2} \right) \Rightarrow \theta = \pm \left( \frac{\pi}{2} - \frac{\theta}{2} \right) (+2k\pi)$$

$$\Rightarrow \theta = \frac{\pi}{3} \text{ et } (\theta = -\pi) \rightarrow \text{à éliminer.}$$

$$M_i \text{ maxi} = -Pa \cos \frac{\pi}{6} (\sin \frac{\pi}{6} + 1) = -\frac{3\sqrt{3}}{4} Pa$$

$U_i$  et  $M_i$  ne sont pas forcément maxi dans la même section.

pour une section circulaire:

$$\tau_{\text{MAX}} = \frac{2}{\pi r^3} W_{i \text{ max}} \quad (\text{car } \tau = \frac{\tau U}{I_p} = \frac{2}{\pi r^3} U)$$

$$\tau_{\text{MAXi}} = \frac{-4Pa}{\pi r^3} \leq R'$$

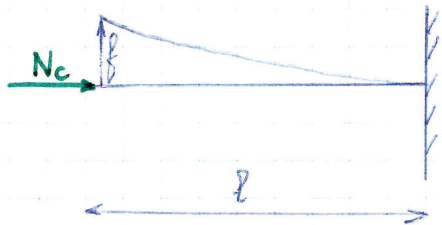
$$\Rightarrow r \geq \sqrt[3]{\frac{-4Pa}{\pi R'}}$$

$$\sigma = \frac{M_z}{I_y} = \frac{4}{\pi r^3} M_{i \text{ max}} \quad (\text{section circulaire})$$

$$\Rightarrow \sigma_{\text{MAXi}} = \frac{4}{\pi r^3} M_{i \text{ MAXi}}$$

$$\sigma_{\text{MAXi}} = \frac{4}{\pi r^3} \left( \frac{|M| + \sqrt{M^2 + U^2}}{2} \right) = \frac{-3\sqrt{3} Pa}{\pi r^3} \leq R \Rightarrow r \geq \sqrt[3]{\frac{-3\sqrt{3} Pa}{\pi R}}$$

### EX 1 CHAP 11



1) Calcul à la compression

$$\sigma = \frac{N}{S} \quad |\sigma| < R_c$$

2) Calcul de  $N_c$

$$N_c < N_{\text{MAXi}}$$

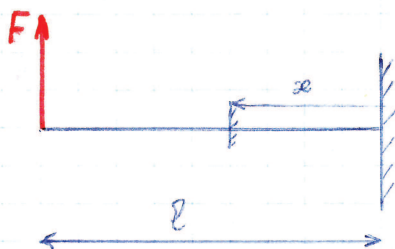
$$\lambda = ax^2 + bx + c \quad ( \text{déformée } N \text{ parabolique} )$$

$$\text{car } \frac{\pi^2 EI}{4l^2} \text{ et } e = 0 \frac{\pi^2 EI}{4l^2}$$

$$\lambda = ax^2 \Rightarrow \frac{\pi^2 EI}{l^2} \frac{d^2 \lambda}{dx^2} = 2ax \quad \frac{\pi^2 EI}{l^2} \frac{d^2 \lambda}{dx^2} = 2a$$

$$\text{eq (2)} \Rightarrow |N_c| = \frac{EI}{l^2} \frac{\int_0^l \left( \frac{d^2 \lambda}{dx^2} \right)^2 dx}{\int_0^l \left( \frac{d\lambda}{dx} \right)^2 dx} = EI \frac{\int_0^l 4a^2 dx}{\int_0^l 4a^2 x^2 dx} = EI \frac{\int_0^l dx}{\int_0^l x^2 dx}$$

$$|N_c| = \frac{3lEI}{l^3} = \frac{3EI}{l^2} \Rightarrow |N_c| = \frac{3EI}{l^2}$$



$$M = -F(l-x)$$

Recherche de la déformée approchée.

$$\frac{d^2 \lambda}{dx^2} = \frac{-M}{EI} = \frac{F(l-x)}{EI}$$

$$\Rightarrow \frac{d\lambda}{dx} = \frac{-F(l-x)^2}{2EI} + k_1$$


$$\underline{CL} \quad x=0 \quad \begin{cases} \lambda=0 \\ \frac{d\lambda}{dx}=0 \end{cases} \Rightarrow k_1 = 0 + \frac{Fl^2}{2EI}$$


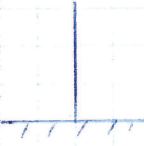

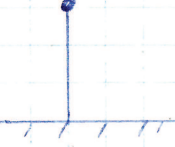

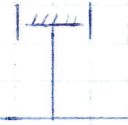
Recherche de  $|N_c|$

$$|N_c| = EI \frac{\int_0^l \left(\frac{d^2\lambda}{dx^2}\right)^2 dx}{\int_0^l \left(\frac{d\lambda}{dx}\right)^2 dx} = EI \frac{\int_0^l (l-x)^2 dx}{\int_0^l \left[-\frac{(l-x)^2}{2} + \frac{Fl^2}{2}\right]^2 dx}$$

$$|N_c| = EI \frac{\frac{l^3}{3}}{\frac{1}{4} \int_0^l [4l^2 - 4lx + 4x^2 + 4l^2 - 4Flx + 4Fl^2]^2 dx} = EI \frac{\frac{l^3}{3}}{\frac{1}{4} \int_0^l (8l^2 - 8lx + 4x^2 - 8Flx + 4Fl^2)^2 dx}$$

$$|N_c| = \left(\frac{1}{4}\right)^{-1} EI \frac{\frac{l^3}{3}}{\frac{l^5}{4} \left(\frac{4}{3} + 1 - 1 + \frac{1}{5} + 4\right)} = \cancel{EI \frac{l^3}{3}} \cdot \frac{4}{l^2} \cdot \frac{1}{8} = EI \cdot \frac{2,5}{l^2}$$

$\lambda$	$ N_c $
parabole	$\frac{3EI}{l^2}$
	$\frac{2,5EI}{l^2}$
Euler $\rightarrow$	$2,467 \frac{EI}{l^2}$

	$ N_c $		$ N_c $
	$\frac{\pi^2 EI}{l^2}$		$\frac{\pi^2 EI}{4l^2}$
	$\frac{4\pi^2 EI}{l^2}$		$\frac{2,05 \pi^2 EI}{l^2}$
	$\frac{9\pi^2 EI}{l^2}$		$\frac{4\pi^2 EI}{l^2}$

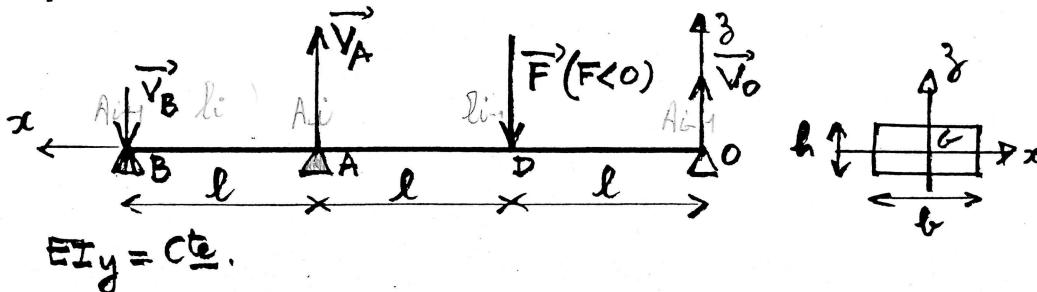
THE END!!



## CONTROLE DE CONNAISSANCES

R. D. M.

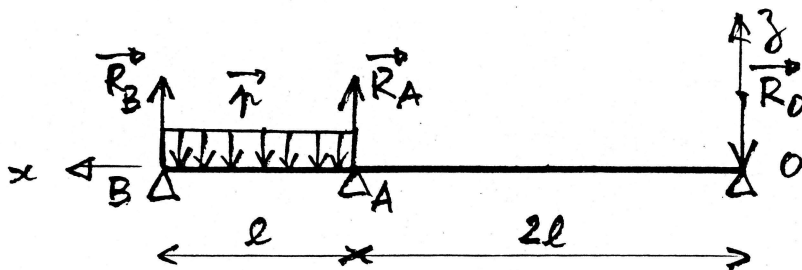
I - Un problème réel peut être schématisé de la façon suivante :



La résolution a donné les résultats suivants:  $V_O = -\frac{3F}{8}$ ;  $V_A = -\frac{7F}{8}$ ;  $V_B = \frac{F}{4}$

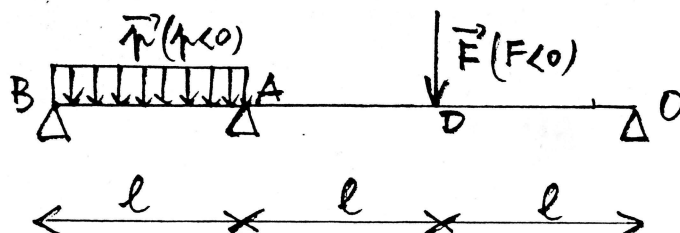
- 1) Tracer le diagramme du moment de flexion de la poutre OB
- 2) Retrouver les valeurs des moments sur appuis par le théorème des trois moments
- 3) Quelle est la section dangereuse? Dimensionner l'épaisseur  $h$  de la poutre en fonction de  $F$ ,  $l$ ,  $b$  et  $R_e$  taux de sécurité à l'élongation ( $\text{daN.mm}^{-2}$ )

II - La même poutre est soumise à une charge répartie, de taux linéaire  $p < 0$  suivant le schéma :



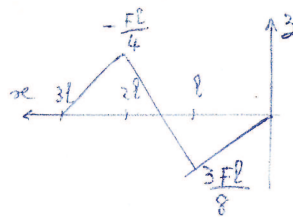
- 1) Equations de la statique en O
- 2) Moments de flexion dans les 2 zones d'étude
- 3) Appliquer la méthode de la ligne élastique pour obtenir une relation reliant  $R_O$ ,  $R_A$ ,  $p$  et  $l$ .
- 4) En déduire les trois actions de contact en O, A et B.
- 5) Sachant que dans l'étude du système du I, on observe une flèche

$\lambda = \frac{5Fl^3}{48EI_y}$  au droit de la force  $\vec{F}$ , quelle flèche trouverait-on au même point si on superposait les 2 charges selon le schéma :



A quelle condition cette flèche résultante peut-elle être nulle?

$$I) \left. \begin{aligned} x \in (0, 2l) \quad M &= \frac{3Fx}{8} \\ x \in (2l, 3l) \quad M &= -\frac{F}{8}(5x-8l) \\ x \in (2l, 3l) \quad M &= -\frac{F}{4}(3l-x) \end{aligned} \right\}$$

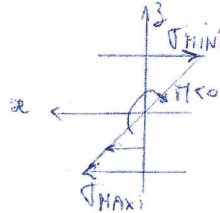


$$\left. \begin{aligned} -\frac{F}{2} \uparrow \quad \downarrow F \quad \uparrow \frac{F}{2} \\ \mu &= \frac{F}{2}x \\ \mu &= \frac{F}{2}(2l-x) \end{aligned} \right\}$$

$$\sum M \Rightarrow 6l M_A = -\frac{6}{2l} \left[ \int_0^{2l} \frac{Fx^2}{2} dx + \int_{2l}^{3l} \frac{Fx}{2}(2l-x) dx \right] \Rightarrow M_A = -\frac{Fl}{4}$$

Calcul de h  $\sigma_{\text{MAXI}} = \frac{M_{\text{maxi}} z_{\text{mini}}}{I_y}$  on  $x=l$

$$\Rightarrow h \geq \frac{3}{2} \sqrt{\frac{-Fl}{6Re}}$$



$$II) \begin{cases} R_0 + R_A + R_B + pl = 0 \\ R_A 2l + 3l R_B + \frac{5}{2} pl^2 = 0 \end{cases}$$

$$x \in (0, 2l) \Rightarrow M = -R_0 \cdot x$$

$$x \in (2l, 3l) \rightarrow M = -R_0 x - R_A(x-2l) - \frac{p(x-2l)^2}{2}$$

ligne élastique

$$EI \lambda = R_0 \frac{x^3}{6} + k_1 x + k_2$$

$$EI \lambda = R_0 \frac{x^3}{6} + R_A \frac{(x-2l)^3}{6} + \frac{p}{24} (x-2l)^4 + k_3 x + k_4$$

$$\underline{cc} \quad \begin{cases} x=0 \quad (\lambda=0) \\ x=2l \quad \begin{cases} \lambda_D=0 \\ \lambda_G=\lambda_D=0 \\ \left(\frac{d\lambda}{dx}\right)_D = \left(\frac{d\lambda}{dx}\right)_G \end{cases} \end{cases} \Rightarrow \begin{cases} k_2 = k_4 = 0 \\ k_1 = k_3 = -\frac{2R_0 l^2}{3} \end{cases}$$

$$\rightarrow 60R_0 + 4R_A + pl = 0$$

Action contact.

$$\text{eq stat} + 60R_0 + 4R_A + pl = 0 \Rightarrow \begin{cases} R_0 = \frac{pl}{48} \\ R_A = -\frac{9pl}{16} \\ R_B = -\frac{11pl}{24} \end{cases}$$

Calcul de  $\lambda$  point  $x=l$ .

2 systèmes superposés (I+II)

$$\begin{cases} \text{syst I} & \lambda_1 = \frac{5Fl^3}{48EI} \\ \text{syst II} & \lambda_2 = \frac{pl}{48} \left(\frac{x}{6}\right)^3 - \frac{pl^3}{72} x \end{cases}$$

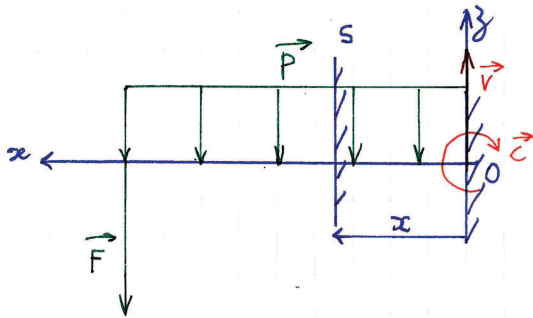
$$k_1 = -\frac{pl^3}{72}$$

$$\text{si } x=l \quad \lambda_2 = -\frac{pl^4}{96EI}$$

$$\lambda = \lambda_1 + \lambda_2 = -\frac{pl^3}{72} - \frac{pl^4}{96EI} + \frac{5Fl}{48EI} \Rightarrow \begin{cases} \lambda = 0 \\ \text{si } F = \frac{pl}{10} \end{cases}$$

# TD RDM

Correction devoir RDM.



$$F = pl.$$

1) Calcul des expressions de M et T.

$$x \in [0, l]$$

$$M = \underbrace{-F(l-x)}_{+} - \underbrace{p \frac{(l-x)^2}{2}}_{+} = -p(l-x) \left[ l + \frac{l-x}{2} \right].$$

$$M = -p(l-x) \frac{(3l-x)}{2}$$

$$T = F + p(l-x)$$

$$T = p(2l-x)$$

2°) Action de contact

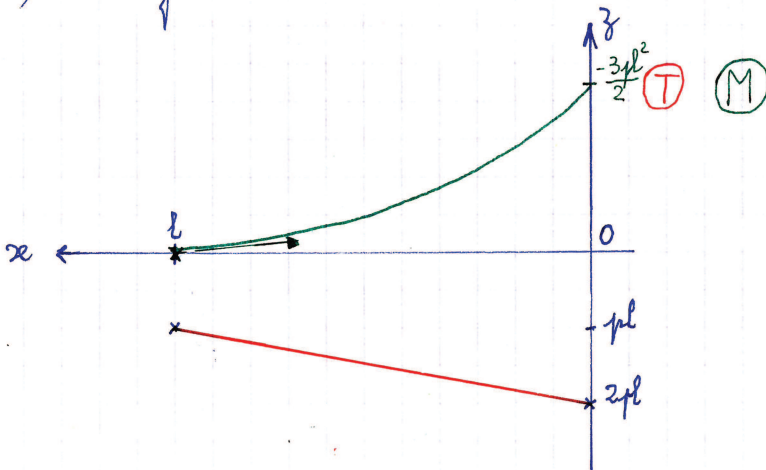
$$\text{en } x=0 \quad M = -C \quad (-F \text{ ext.})$$

$$C = \frac{3pl^2}{2}$$

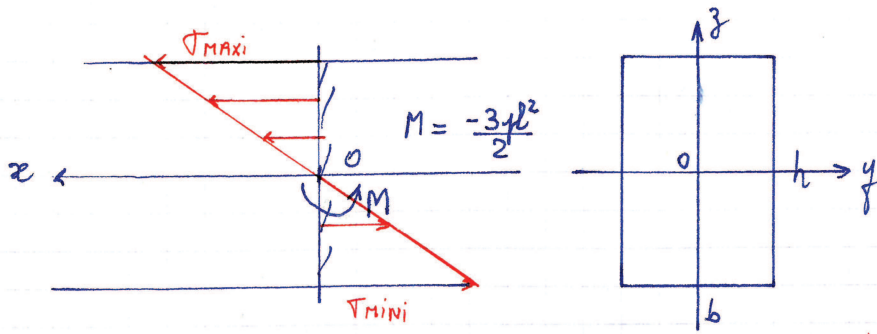
$$T = -V$$

$$V = -2pl$$

3) Diagrammes:

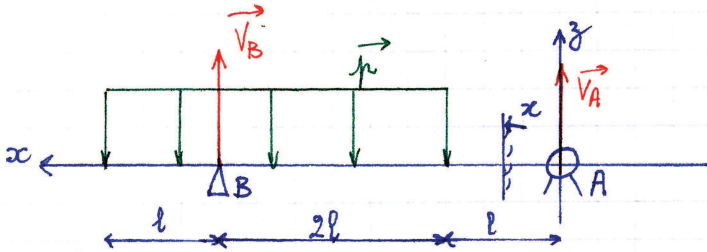


4) Calcul de la contrainte normale  $\sigma$  dans la section d'encastrement.



$$\sigma = \frac{Mz}{I_y} \Rightarrow \frac{-3pl^2 h \cdot 12}{2 \cdot 2 \cdot b \cdot h^3} = \frac{-9pl^2}{bh^2} = \sigma_{\text{maxi}}$$

II)



1) Calcul des actions de contact.

$$V_A + V_B + 3pl = 0$$

$$\sum \mathcal{M}_A^+ = -V_B \cdot 3l - 3pl \cdot \frac{5l}{2} = 0$$

$$V_B = -\frac{5pl}{2}$$

$$V_A = -\frac{pl}{2}$$

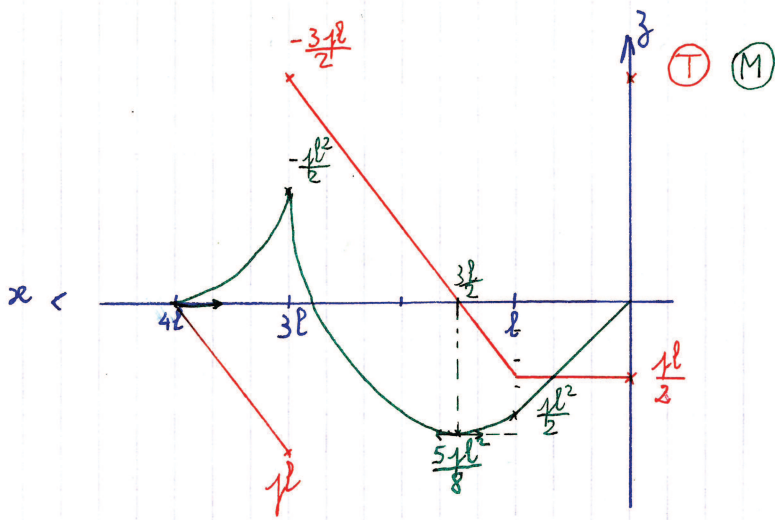
2) Expression de M et T.

$$x \in (0, l) \quad \begin{cases} T = -V_A = \frac{pl}{2} \\ M = -V_A \cdot x = \frac{plx}{2} \end{cases}$$

$$x \in (l, 3l) \quad \begin{cases} T = -[V_A + p(x-l)] \\ T = \frac{pl}{2} (3l - 2x) \\ M = -[V_A \cdot x + \frac{p(l-x)^2}{2}] \\ M = \frac{plx}{2} - \frac{p}{2}(x-l)^2 \\ M = \frac{p}{2}(3lx - x^2 - l^2) \end{cases}$$

$$x \in (3l, 4l) \quad \begin{cases} T = p(4l - x) \\ M = -p \frac{(4l - x)^2}{2} \end{cases}$$

3) Diagramme de M et T.

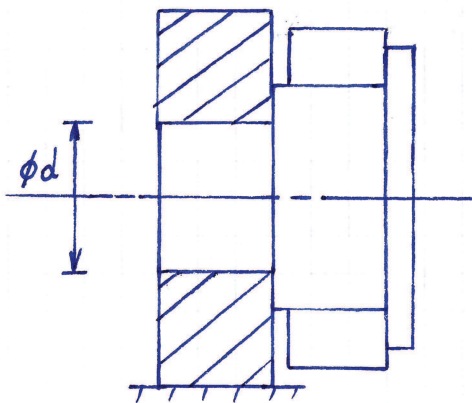


3) Section où  $T_{\text{MAX}}$ .

$$|M|_{\text{maxi}} = \frac{-5pl^2}{8} \quad \text{on a } T=0 \Rightarrow x = \frac{3l}{2}$$

4) Calcul de  $\tau_{\text{MAXI}}$

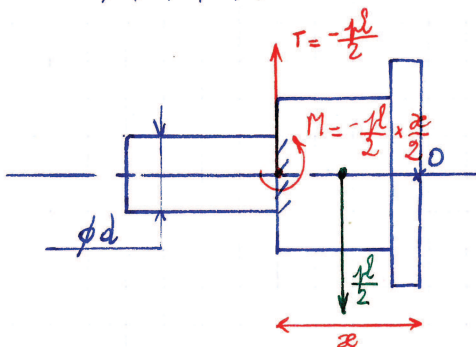
$$\tau = \frac{T}{S_z} \Rightarrow \tau_{\text{MAXI}} = \frac{T_{\text{MAXI}}}{S_z} = \frac{-3pl}{2} \times \frac{3}{2a^2} = \frac{-9pl}{4a^2} = \tau_{\text{MAXI}}$$



Action de l'axe sur la poutre

$$V_0 = \frac{-pl}{2}$$

$$\text{Action de la poutre / axe} = \frac{pl}{2}$$



$$\tau_{\text{MAXI}} = \frac{|T|_{\text{MAXI}}}{S_z} = \frac{-pl \cdot 4 \times 4}{2 \times 3\pi d^2}$$

$$\Rightarrow \tau_{\text{MAXI}} = \frac{-8pl}{3\pi d^2}$$

condition de sécurité :

$$|\tau|_{\text{MAX}} \leq R_g \Rightarrow \frac{-8pl}{3\pi d^2} \leq R_g \Rightarrow d \geq \sqrt{\frac{-8pl}{3\pi R_g}}$$